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Residual Representations of Spacetime

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Spacetime is modelled by binary relations—by the classes of the automorphisms $\mathbf{GL}(\mathbb{C}^2)$ of a complex two-dimensional vector space with respect to the definite unitary subgroup $\mathbf{U}(2)$. In extension of Feynman propagators for particle quantum fields representing only the tangent spacetime structure, global spacetime representations are given, formulated as residues using energy–momentum distributions with the invariants as singularities. The associated quantum fields are characterized by two invariant masses—for time and position—supplementing the one mass for the definite unitary particle sector with another mass for the indefinite unitary interaction sector without asymptotic particle interpretation.

1. INTRODUCTION

Quantum theory starts with operations (Finkelstein, 1996). An experiment for quantum structures probes a "diagonalization" of the operator under question, for example, of a time and position translation, of a rotation, or of a charge transformation, with the eigenvalues as possible experimental results, for example, energy and momenta, mass or spin, or a charge number, respectively. Therewith, I shall take the radical point of view that all relevant mathematical structures and tools used in quantum theories have to have an interpretation in terms of operations, of monoids, groups, and algebras, especially of real Lie groups and Lie algebras, realized and represented as acting upon sets, especially upon complex vector spaces with a reality-defining conjugation. Representation theory gives the irreducible and—for linear structures—also the nondecomposable action spaces. Almost all functions relevant for physics can be interpreted as arising from representation structures (Vilenkin and Klimyk, 1991).

Physical events represent spacetime operations, for example, translations, rotations, and boosts. A quantum-mechanical dynamics, implemented by iH (Hamiltonian H with eigenvalues $E \in \mathbb{R}$) as basis for the time-translation Lie algebra $\mathbb{R}$, is a representation of the causal time group $\mathbf{D}(1) = \exp \mathbb{R}$, irreducible

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for $e^{itE} \in \mathbf{U}(1)$, for example, for the harmonic oscillator or for creation and annihilation operators in quantum particle fields. In the Schrödinger picture the time representations in $\mathbf{U}(1)$ are realized on a Hilbert space with the scalar product (probability amplitudes) induced by the time representing $\mathbf{U}(1)$. The wave functions come as position-translation-representation matrix elements, for example, the scattering- and bound-state wave functions $\psi(r)$ in rotation symmetric problems with $r\psi(r) \sim e^{\pm ir|Q|}, e^{-r|Q|}$ as compact $\mathbf{U}(1)$ and noncompact $\mathbf{D}(1)$-representations respectively of the radial translation monoid $\mathbb{R}^+$. In quantum mechanics the time translation eigenvalue iE (energy E) and the position translation eigenvalue Q are in a unique correspondence: For example, for a constant potential V_0 with $-\frac{Q^2}{2} = E - V_0$, the scattering case is given by $E > V_0$ with imaginary eigenvalues $\pm i|Q|$ and momentum $|Q|$ whereas the bound states come with $E < V_0$ where $|Q|$ cannot be interpreted as momentum.

In analogy to the dynamics for time $\mathbf{D}(1) = \exp \mathbb{R}$, the representations[2] of the globally symmetric manifold $\mathbf{D}(2) = \exp \mathbb{R}^4$ as spacetime model (see Saller, 1997, 1999; also the detailed discussion given later), with the Minkowski translations as tangent space $\mathbb{R}^4$, will be considered as possible candidates for a spacetime dynamics:

$$\text{time dynamics: } \mathbf{rep}\, \mathbf{D}(1) \quad \text{with}\, \mathbf{D}(1) = \mathbf{GL}(\mathbb{C})/\mathbf{U}(1),$$

$$\text{spacetime dynamics: } \mathbf{rep}\, \mathbf{D}(2) \quad \text{with}\, \mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2).$$

The spacetime manifold $\mathbf{D}(2) = \mathbf{D}(\mathbf{1}_2) \times \mathbf{SD}(2)$ contains, as factor for the causal group $\mathbf{D}(1)$, the rank 1 position manifold $\mathbf{SD}(2) \cong \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$ with another Cartan subgroup $\mathbf{SO}_0(1, 1) \cong \exp \mathbb{R}$. An independent realization of both factors in the Cartan subgroups $\mathbf{D}(1) \times \mathbf{SO}_0(1, 1)$ of the rank 2 spacetime manifold $\mathbf{D}(2)$ is characterized by two continuous invariants.

For particles with mass m, the energy–momenta $(q_0, \vec{q})$ as eigenvalues for spacetime translations $(x_0, \vec{x})$ are on shell, that is $q^2 = m^2$. With Wigner (1939) we know that particle quantum fields implement definite unitarily the spacetime translation invariant with the mass $m^2 = q_0^2 - \vec{q}^2$ as the translation eigenvalue. In the following the off shell structures of a propagator, that is for $q^2 \neq m^2$, will be extended for a complete realization of rank 2 spacetime $\mathbf{D}(2)$ with its two noncompact invariants.

Representation matrix elements[3] of a real Lie group are analytic functions on this group

$$D : G \to \mathbb{C}, \quad g(x) \mapsto D(x)$$

for example, $\frac{\vec{x}}{r} i \sin r$ for compact spin $\mathbf{SU}(2)$ or $\cos xm$ for compact axial rotations $\mathbf{U}(1)$ or both $\cos tm$ and $t \cosh tm$ for noncompact time $\mathbf{D}(1)$. According to the

[2] **irrep** G and **rep** G denotes the (irreducible) representation classes of a group G.

[3] In the following the short "representation" can stand for the more correct "representation matrix element(s)."

Peter–Weyl theorem (Folland, 1995; Peter and Weyl, 1927), the span of the irreducible representation matrix elements of a compact Lie group is dense in the continuous functions on this group.

In a harmonic analysis, representation matrix elements of a group can be written as Fourier transforms of distributions of their Lie algebra forms, for example, of energies or angular momenta values, where the representation characterizing invariants come as singularities, that is as poles of the distributions. This defines the concept of residual representations. In the following, familiar algebraic representation concepts (Helgason, 1978), such as weights, invariants, and Lie algebras are translated into the language of residual representations.

In analogy to Lie groups such as the compact $\mathbf{U}(n)$ or the noncompact $\mathbf{D}(1)$, symmetric spaces such as the noncompact position manifold $\mathbf{SD}(2)$ and spacetime $\mathbf{D}(2)$ also have linear representations that will be considered in analogy to the representations of the time group $\mathbf{D}(1)$. To construct residual representations of the rank 2 spacetime manifold $\mathbf{D}(2)$ distributions of the energy–momenta $q \in \mathbb{R}^4$ (tangent space forms) are used, supported by two invariant masses $q^2 \in \{m_0^2, m_3^2\}$ characterizing the Cartan subgroup $\mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}_0(1, 1)$-representations for time and position.

2. QUANTUM REPRESENTATIONS OF TIME

A dynamics is a representation of time (translation), realized in quantum mechanics by the quantization (anti-) commutators of the quantum-algebra-generating operators. In the simplest cases of a harmonic oscillator with Hamiltonian $H = \frac{\mathbf{p}^2}{2M} + m^2 M \frac{\mathbf{x}^2}{2}$ for mass M and frequency m, or of a free mass point with $H = \frac{\mathbf{p}^2}{2M}$ for frequency $m \to 0$, the time-dependent commutation relations of the dual quantum algebra generating position–momentum pair $(\mathbf{x}, \mathbf{p})$ give the time representation matrix elements

$$\mathbf{D}(1) \ni e^t \mapsto D(t) = \begin{pmatrix} [i\mathbf{p}, \mathbf{x}] & [\mathbf{x}, \mathbf{x}] \\ [\mathbf{p}, \mathbf{p}] & [\mathbf{x}, -i\mathbf{p}] \end{pmatrix}(t)$$
$$= \begin{cases} \begin{pmatrix} \cos tm & \frac{i}{Mm} \sin tm \\ iMm \sin tm & \cos tm \end{pmatrix} & \in \mathbf{SO}(2) \\ \begin{pmatrix} 1 & \frac{it}{M} \\ 0 & 1 \end{pmatrix} & \in \mathbf{U}(1, 1) \end{cases}$$

with the shorthand notation $[a(s), b(t)]_\epsilon = [a, b]_\epsilon(t - s)$, $\epsilon = \pm 1$, valid for all matrix elements. Those representations arise from the complex irreducible and nondecomposable time representations with creation and annihilation operator (u, u*) and nil- and eigenoperators (Saller, 1989) (b, g, b$^\times$, g$^\times$) respectively:

$$\mathbf{D}(1) \ni e^t \mapsto \begin{cases} [\mathrm{u}^\star, \mathrm{u}]_\epsilon(t) = e^{itm} & \in \mathbf{U}(1), \\ \begin{pmatrix} [\mathrm{g}^\times, \mathrm{b}]_\epsilon & [\mathrm{b}^\times, \mathrm{b}]_\epsilon \\ [\mathrm{g}^\times, \mathrm{g}]_\epsilon & [\mathrm{b}^\times, \mathrm{g}]_\epsilon \end{pmatrix}(t) = \begin{pmatrix} 1 & it\nu \\ 0 & 1 \end{pmatrix} e^{itm} & \in \mathbf{U}(1, 1). \end{cases}$$

The quantization opposite commutators implement the Lie algebra of the basic space endomorphisms, for example, the Hamiltonians mentioned previously. For the harmonic oscillator the $\mathbf{U}(1)$-induced Fock form $\langle \cdots \rangle_{\mathrm{F}}$ of the time dependent anticommutators arises as time derivative of the quantization

$$\begin{pmatrix} \langle\{i\mathbf{p}, \mathbf{x}\}\rangle_{\mathrm{F}} & \langle\{\mathbf{x}, \mathbf{x}\}\rangle_{\mathrm{F}} \\ \langle\{\mathbf{p}, \mathbf{p}\}\rangle_{\mathrm{F}} & \langle\{\mathbf{x}, -i\mathbf{p}\}\rangle_{\mathrm{F}} \end{pmatrix}(t) = \begin{pmatrix} i \sin tm & \frac{1}{Mm} \cos tm \\ Mm \cos tm & i \sin tm \end{pmatrix} = \frac{1}{im} \frac{d}{dt} D(t)$$

For the general quantum mechanical case with $iH = i[\frac{\mathbf{p}^2}{2M} + V(\mathbf{x})]$ as basis for the represented Lie algebra[4] $\log \mathbf{D}(1) \cong \mathbb{R}$, the time $\mathbf{D}(1)$-representation matrix elements as the ground state values $\langle [a(s), b(t)]_\epsilon \rangle = \langle [a, b]_\epsilon \rangle (t - s)$ of the position–momentum commutators can be computed from the imaginary and time translation antisymmetric position commutator

$$\langle [\mathbf{x}, \mathbf{x}] \rangle (t) = \int_0^\infty dm^2 \, \mu(m^2) i \, \frac{\sin tm}{Mm}$$

with a spectral measure $\mu(m^2)$ for the time translation eigenvalues $m \in \mathbb{R}$ (frequencies, energies), for example, $\mu(m^2) = \delta(m^2 - m_0^2)$ with $m_0^2 > 0$ for oscillator and $m_0 = 0$ for free mass point, and $\mathbf{p} = M \frac{d\mathbf{x}}{dt}$

$$\left\langle \begin{pmatrix} [i\mathbf{p}, \mathbf{x}] & [\mathbf{x}, \mathbf{x}] \\ [\mathbf{p}, \mathbf{p}] & [\mathbf{x}, -i\mathbf{p}] \end{pmatrix} \right\rangle (t)$$

$$= \int_0^\infty dm^2 \, \mu(m^2) \begin{pmatrix} \cos tm & \frac{i}{Mm} \sin tm \\ iMm \sin tm & \cos tm \end{pmatrix} \in \mathbf{rep}\, \mathbf{SO}(2)$$

In the case of a compact time development, that is, representations in $\mathbf{U}(1)$ or $\mathbf{SO}(2)$, where there exists a basis of normalizable energy eigenvectors (for the oscillator build by the monomials of creation and annihilation operator), the energy measure is definite $\mu(m^2) \geq 0$.

[4] The Lie group to Lie algebra transition $G \mapsto \log G$ is denoted with the logarithm log as covariant functor.

3. TIME AND POSITION TRANSLATIONS

3.1. The Lie Groups for the Translations

Translations are formalized by additive groups (vector spaces) $\mathbb{R}^n$. It will be convenient to introduce a distinguishing notation for the Lie group and the Lie algebra involved, which have an isomorphic Abelian Lie group structure

$$\left.\begin{array}{lll}\text{Lie group} & \mathbf{D}(1) = \exp\mathbb{R} & = \{e^x \mid x \in \mathbb{R}\} \\ \text{Lie algebra} & \mathbb{R} & = \log\mathbf{D}(1) = \{x \mid x \in \mathbb{R}\}\end{array}\right\}, \quad \exp\mathbb{R} \cong \mathbb{R}$$

The noncompact group $\mathbf{D}(1)$ as universal covering group is locally isomorphic to the compact one $e^{i\alpha} \in \mathbf{U}(1) = \exp i\mathbb{R} \cong \mathbb{R}/\mathbb{Z}$ with Lie algebra $\log\mathbf{U}(1) = i\mathbb{R}$.

The groups $\mathbf{U}(1)$ and $\mathbf{D}(1)$ are, as real one-dimensional Lie groups, isomorphic to the axial rotations $\mathbf{SO}(2)$ and the Procrustes[5] dilatation group $\mathbf{SO}_0(1, 1)$ respectively, that is the one-dimensional boosts

$$\begin{array}{ll}\text{compact} & \mathbf{U}(1) \cong \mathbf{SO}(2) \quad = \left\{\begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix} \middle| \alpha \in \mathbb{R}\right\} \\ \text{noncompact} & \mathbf{D}(1) \cong \mathbf{SO}_0(1, 1) = \left\{\begin{pmatrix}\cosh x & \sinh x \\ \sinh x & \cosh x\end{pmatrix} \middle| x \in \mathbb{R}\right\}\end{array}$$

Those orthogonal groups with invariant bilinear forms of the two-dimensional vector space they are acting upon, will be called self-dual representations[6] of $\mathbf{U}(1)$ and $\mathbf{D}(1)$ respectively with the obvious isomorphy (for $\mathbf{SO}(2)$ only in the complex)

$$\begin{array}{ll}\text{definite unitary: } \mathbf{SO}(2) & \ni \begin{pmatrix}\cos\alpha & i\ \sin\alpha \\ i\sin\alpha & \cos\alpha\end{pmatrix} \cong \begin{pmatrix}e^{i\alpha} & 0 \\ 0 & e^{-i\alpha}\end{pmatrix} \in \mathbf{SU}(2) \\ \text{indefinite unitary: } \mathbf{SO}_0(1, 1) & \ni \begin{pmatrix}\cosh x & \sinh x \\ \sinh x & \cosh x\end{pmatrix} \cong \begin{pmatrix}e^{x} & 0 \\ 0 & e^{-x}\end{pmatrix} \in \mathbf{SU}(1, 1)\end{array}$$

3.2. Real Operations Have Unitary Representations

The algebraic and topological completeness of the complex field $\mathbb{C}$ allows the definition of the transcendental number e involving "exponential completeness"

[5] Procrustes in the Greek mythology either shrinked or stretched his visitors—tall or short respectively—to death.

[6] For a group and a Lie algebra dual representations on finite dimensional dual vector spaces are related to each other by inverse and negative transposition respectively.

$\exp \mathbb{C} = \mathbb{C} \setminus \{0\}$ and, therewith, the exponential transition from local linear structures (tangent vector spaces, Lie algebras) to global possibly nonlinear structures (symmetric spaces, Lie groups). Therefore, I will consider representations on complex vector spaces only. The complex representations of the physically arising only real Lie groups or Lie algebras have to be unitary, definite $\mathbf{U}(n)$ or indefinite $\mathbf{U}(p, q)$, in order to recognize the realness also in the representation. Therewith, the complex numbers are always used together with the canonical conjugation, that is as the doubled real field $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$.

Only for one complex dimension unitarity is unique, characterized by the real Lie group $\mathbf{U}(1) = \exp i\mathbb{R}$. The n unitarities for n complex dimensions go with the signature: For example, in two dimensions the $\mathbf{U}(2)$-conjugation of 2×2-matrices can be written as the familiar conjugate transposition that exchanges the elements of the skewdiagonal whereas the $\mathbf{U}(1, 1)$-conjugation can be written with an exchange of the diagonal elements

$$\mathbf{U}(2)\text{-conjugation: } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \leftrightarrow \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix}$$

$$\mathbf{U}(1, 1)\text{-conjugation: } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \leftrightarrow \begin{pmatrix} \overline{\delta} & \overline{\beta} \\ \overline{\gamma} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3.3. Nildimensions for Noncompact Groups

Noncompact groups have reducible but nondecomposable representations (Boerner, 1955; Saller, 1989; Shelobenko, 1958, 1959) where the representation space cannot be spanned by eigenvectors only—there occur also nilvectors, that is principal vectors that are not eigenvectors. The linear operators involved have a Jordan triangular form with nontrivial off-diagonal entries.

The situation is characterized by the nondecomposable representations of the group $\mathbf{D}(1)$ with an eigenvalue m for $e^x \mapsto e^{ixm}$, which comes multiplied with an automorphism of the representation space $V \cong \mathbb{C}^{1+N}$ and can be written with a nilcyclic matrix M_N (nil-Hamiltonian), nilpotent to the power $N + 1$

$$\mathbf{D}(1) \ni e^x \mapsto e^{ix(m+M_N)} \cong e^{ixm} \begin{pmatrix} 1 & ix & \frac{(ix)^2}{2!} & \cdots & \frac{(ix)^N}{N!} \\ 0 & 1 & ix & \cdots & \frac{(ix)^{N-1}}{(N-1)!} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & ix \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(\mathbb{C}^{1+N})$$

$$(M_N)^N \neq 0,\ (M_N)^{N+1} = 0,\ M_N \cong \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The natural number N is called the nildimension with $1 + N$ the dimension of the nondecomposable representation. Irreducible representations have trivial nildimension $N = 0$ and $M_0 = 0$. For $N \geq 1$ the conjugation is indefinite, that is the group image is a subgroup of **U**(1, 1), **U**(2, 1), **U**(2, 2), etc.

An example for nontrivial nildimensions in quantum mechanics is the radial part ψ_{nL} of the bound state wave functions in the hydrogen atom: It is a linear combination of matrix elements $r \mapsto r^N e^{-\frac{r}{k}}$ of noncompact representations of the radial translations with eigenvalue $-\frac{1}{k}$, $k = n + L + 1$

$$\mathbb{R}^+ \ni r \mapsto D_{nL}(r) = r\psi_{nL}(r) \sim \left(\frac{2r}{k}\right)^{L+1} \mathcal{L}_n^{2L+1}\left(\frac{2r}{k}\right) e^{-\frac{r}{k}}$$

with the Laguerre polynomials $\mathcal{L}$ as combinations of radial powers r^N.

An example for nontrivial nildimensions in quantum field theory is quantum electrodynamics where the nonparticle components of the **U**(1)-gauge field, which come in addition to the left and right circularly polarized particle degrees of freedom (photons), that is the Coulomb force inducing degree of freedom and the so-called gauge degree of freedom, are spacetime translation nilvectors (Saller, 1993, 1995), that is principal vectors that are no eigenvectors. The dichotomy between particles and interaction degrees of freedom in the electromagnetic potential reflects the compact and noncompact Cartan subgroups in the Lorentz group $\mathbf{SO}(2) \times \mathbf{SO}_0(1, 1) \subset \mathbf{SO}_0(1, 3)$, represented definite unitarily $\mathbf{SO}(2) \to \mathbf{U}(2)$ for the photons and indefinite unitarily $\mathbf{SO}_0(1, 1) \to \mathbf{U}(1, 1)$ for Coulomb and gauge degree of freedom. The nilpotency of the BRS-generator (Becchi *et al.*, 1976) with the power $N + 1 = 2$ has its origin in the time translation representation $\mathbf{D}(1) \to \mathbf{U}(1, 1)$ for the two nonparticle degrees of freedom with $M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the nil-Hamiltonian that fulfills $M_1^2 = 0$.

4. THE SPACETIME REPRESENTATION STRUCTURE OF QUANTUM PARTICLE FIELDS

Particle fields are appropriate for describing free particles; they implement definite unitary representations of the Poincaré Lie algebra (Mackey, 1968; Wigner, 1939) $\log \mathbf{SO}_0(1, 3) \vec{\oplus} \mathbb{R}^4$.

A particle field, in the simplest case a hermitian scalar massive field $\mathbf{\Phi}$, $m > 0$, with creation and annihilation operators $(\mathrm{u}, \mathrm{u}^\star)$

$$\mathbf{\Phi}(x) = \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{m}{q_0}}\, \frac{e^{ixq}\mathrm{u}(\vec{q}) + e^{-ixq}\mathrm{u}^*(\vec{q})}{\sqrt{2}}, \quad q_0 = \sqrt{m^2 + \vec{q}^2}$$

$$[\mathrm{u}^\star(\vec{p}), \mathrm{u}(\vec{q})] = (2\pi)^3\delta(\vec{q} - \vec{p}) = \langle\{\mathrm{u}^\star(\vec{p}), \mathrm{u}(\vec{q})\}\rangle = \langle \mathrm{u}^\star(\vec{p})\mathrm{u}(\vec{q})\rangle$$

is characterized by its quantization,[7] causally supported and on shell

$$\frac{[\mathbf{\Phi}, \mathbf{\Phi}](x)}{m} = i\,\frac{\mathbf{s}(x \mid m)}{m} = \int \frac{d^4q}{(2\pi)^3}\epsilon(q_0)\delta(q^2 - m^2)\, e^{ixq} = 0 \ \text{ for } \ x^2 < 0$$

and its Feynman propagator adding up the Fock form value of the quantization-opposite commutator, also on shell

$$\frac{\langle\{\mathbf{\Phi}, \mathbf{\Phi}\}\rangle_{\mathrm{F}}(x)}{m} = \frac{\mathbf{C}(x \mid m)}{m} = \int \frac{d^4q}{(2\pi)^3}\delta(q^2 - m^2)\, e^{ixq}$$

and the $\epsilon(x_0)$-multiplied quantization (Gel'fand and Shilov, 1963) which also has off shell contributions, for $q^2 \neq m^2$

$$\frac{\epsilon(x_0)\mathbf{s}(x \mid m)}{m} = \frac{1}{\pi}\int \frac{d^4q}{(2\pi)^3}\, \frac{1}{-q_{\mathrm{P}}^2 + m^2}\, e^{ixq} \ \text{(principal value P)}$$

$$\frac{\langle\{\mathbf{\Phi}, \mathbf{\Phi}\}(x) \pm \epsilon(x_0)[\mathbf{\Phi}, \mathbf{\Phi}](x)\rangle_{\mathrm{F}}}{m} = \frac{\mathbf{C}(x \mid m) \pm i\epsilon(x_0)\mathbf{s}(x \mid m)}{m}$$

$$= \pm\frac{1}{i\pi}\int \frac{d^4q}{(2\pi)^3}\, \frac{1}{q^2 \mp io - m^2}\, e^{ixq}$$

The harmonic contributions in the quantization

$$i\,\frac{\mathbf{s}(x \mid m)}{m} = \int \frac{dq_0}{(2\pi)^2}\, e^{ix_0q_0}\epsilon(q_0)\vartheta\left(q_0^2 - m^2\right)\, \frac{\sin r\sqrt{q_0^2 - m^2}}{r}$$

[7] The linear Minkowski spacetime parametrization is used in the notation for (anti)commutators $[A(y), B(x)]_\pm = [A, B]_\pm(x - y)$.

and in the Feynman propagator

$$\frac{\mathbf{C}(x \mid m)}{m} \pm \frac{\epsilon(x_0) i \mathbf{s}(x \mid m)}{m}$$

$$= \int \frac{dq_0}{(2\pi)^2} e^{ix_0q_0} \begin{cases} \left[\vartheta\left(q_0^2 - m^2\right) \frac{\sin r\sqrt{q_0^2 - m^2}}{r} \right. \\ \pm i\vartheta\left(q_0^2 - m^2\right) \frac{\cos r\sqrt{q_0^2 - m^2}}{r} \\ \left. \pm i\vartheta\left(m^2 - q_0^2\right) \frac{e^{-r\sqrt{m^2 - q_0^2}}}{r} \right] \end{cases}$$

show irreducible (definite unitary) time translation representation matrix elements

$$\mathbb{R} \ni x_0 \mapsto e^{\pm i x_0 q_0} \in \mathbf{U}(1)$$

With the polar coordinate position translation decomposition

$$\vec{x} \in \mathbb{R}^3 \cong \mathbb{R}^+ \times \mathbf{SO}(3)/\mathbf{SO}(2)$$

and the geometrical Kepler factor $\frac{1}{r}$ for the sphere surface $\mathbf{SO}(3)/\mathbf{SO}(2)$-distribution, the position radial translation monoid $r \in \mathbb{R}^+$ is represented by $\frac{\sin r|\vec{q}|}{r}$ (spherical Bessel function) with $\sin r|\vec{q}|$ as matrix element of a compact group for the quantization $\mathbf{s}(x \mid m)$ and the Fock form function $\mathbf{C}(x \mid m)$. In the propagator contribution $\epsilon(x_0)\mathbf{s}(x \mid m)$ there arise the $r = 0$-singular spherical Neumann function $\frac{\cos r|\vec{q}|}{r}$ that contains $\cos r|\vec{q}|$ as a compact position translation representation matrix element. The additional off shell induced Yukawa contributions display a representation matrix element of the radial position translations in a noncompact (indefinite unitary) group

$$\mathbb{R}^+ \ni r \mapsto \begin{cases} e^{\pm ir|\vec{q}|} \in \mathbf{SO}(2) \\ e^{-r|Q|} \in \mathbf{SO}_0(1,1) \end{cases}$$

The off shell contributions with the Yukawa interactions in the Feynman propagator are no definite unitary representation matrix elements.

The time projection $\int d^3x$ of quantization and Feynman propagator gives matrix elements for the representation of time translations in the rest system of a massive particle

$$x_0 \mapsto \int d^3x \begin{pmatrix} \mathbf{C}(x \mid m) \\ i\mathbf{s}(x \mid m) \\ \epsilon(x_0) i\mathbf{s}(x \mid m) \end{pmatrix}$$

$$= \int dE \epsilon(E) \begin{pmatrix} E \\ m \\ m\epsilon(x_0) \end{pmatrix} \delta(E^2 - m^2)\, e^{ix_0E} = \begin{pmatrix} \cos x_0 m \\ i\ \sin x_0 m \\ i\ \sin|x_0|m \end{pmatrix}$$

The analogue position projection $\int dx_0$

$$\vec{x} \mapsto 2\pi \int \frac{dx_0}{im} \begin{pmatrix} i\mathbf{s}(x \mid m) \\ \mathbf{C}(x \mid m) \\ \epsilon(x_0) i\mathbf{s}(x \mid m) \end{pmatrix} = \int \frac{dQ}{2} \begin{pmatrix} 0 \\ 0 \\ \vartheta(Q^2 - m^2) \end{pmatrix} e^{-r|Q|} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{e^{-rm}}{r} \end{pmatrix}$$

is nontrivial only for the off shell contributions with radial translation representation matrix element e^{-rm} in a noncompact group.

Particle fields display in the quantizations $\mathbf{s}(x \mid m)$ and the Fock form $\mathbf{C}(x \mid m)$, both on shell $q^2 = m^2$, matrix elements of definite unitary representations for the translations. The off shell contributions in $\epsilon(x_0)\mathbf{s}(x \mid m)$ involve matrix elements for indefinite unitary representation matrix elements for position translations $\mathbb{R}^3$.

5. HOMOGENEOUS MODELS FOR TIME, POSITION, AND SPACETIME

5.1. Exponentiating Time Translations

The time translations as a real one-dimensional vector space $x_0 = \overline{x_0} \in \mathbb{R}$ are isomorphic (as Lie group) to its exponent $\mathbf{D}(1) = \exp \mathbb{R}$, the time group. They constitute the noncompact part (modulus) of the full complex group, given by the phase classes

$$\text{time: } \mathbf{GL}(\mathbb{C})/\mathbf{U}(1) = \mathbf{D}(1) = \exp \mathbb{R} \cong \mathbb{R}$$

5.2. Exponentiating Position Translations

In the semidirect Euclidean group $\mathbf{SO}(3) \vec{\times} \mathbb{R}^3$, the position translations as a real three-dimensional vector space $\mathbb{R}^3$ are isomorphic—as vector space with rotation action—to the Lie algebra of the rotations $\log \mathbf{SO}(3) \cong \mathbb{R}^3$. In the $\mathbf{SU}(2)$-formulation, the rotations $\mathbf{SO}(3)$ are represented by the adjoint action of its covering group **SU**(2)

$$\mathbf{SO}(3)\,\vec{\times}\,\mathbb{R}^3 \sim \mathbf{SU}(2)\,\vec{\times}\,\mathbb{R}^3, \qquad O.\vec{x} \sim u \circ \vec{x}\vec{\sigma} \circ u^{-1}$$
$$\text{with } \vec{x}\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

$$u \in \mathbf{SU}(2) \Rightarrow O^a_b = \frac{1}{2}\,\mathrm{tr}\,\sigma^a u \sigma^b u^{-1}, \quad O \in \mathbf{SO}(3) \cong \mathbf{SU}(2)/\{\pm \mathbf{1}_2\}$$

In the Pauli representation, the position translations are hermitian 2×2-matrices, that is representatives[8] of the classes of all complex special matrices $\log \mathbf{SL}(\mathbb{C}^2) \cong \mathbb{R}^3 \oplus (i\mathbb{R})^3$ with respect to the special unitary ones $\log \mathbf{SU}(2) \cong (i\mathbb{R})^3$

$$\vec{x}\vec{\sigma} = (\vec{x}\vec{\sigma})^\star \Subset \log \mathbf{SL}(\mathbb{C}^2)/\log \mathbf{SU}(2)$$

The global position manifold arises by exponentiation, isomorphic as symmetric space to the classes of the Lorentz covering group $\mathbf{SL}(\mathbb{C}^2)$ with respect to the rotation covering group **SU**(2)

$$\text{position: } \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SD}(2) = \exp \mathbb{R}^3 \cong \mathbb{R}^3$$

The global symmetric space position **SD**(2) and its tangent vector space $\mathbb{R}^3$ have a manifold isomorphy only, $\exp \mathbb{R}^3 \neq (\exp \mathbb{R})^3$.

5.3. Exponentiating Spacetime Translations

In the Poincaré group $\mathbf{SO}_0(1,3)\,\vec{\times}\,\mathbb{R}^4$ the translations $\mathbb{R}^4$ are not isomorphic to the Lie algebra of the Lorentz group $\log \mathbf{SO}_0(1,3) \cong \mathbb{R}^6$. In the $\mathbf{SL}(\mathbb{C}^2)$-formulation, the Lorentz transformations $\mathbf{SO}_0(1,3)$ are represented by the conjugate adjoint action of its covering group $\mathbf{SL}(\mathbb{C}^2)$

$$\mathbf{SO}_0(1,3)\,\vec{\times}\,\mathbb{R}^4 \sim \mathbf{SL}(\mathbb{C}^2)\,\vec{\times}\,\mathbb{R}^4, \qquad \Lambda\,.\,x \sim s \circ x \circ s^\star$$
$$\text{with } x = x_k\sigma^k = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

$$s \in \mathbf{SL}(\mathbb{C}^2) \Rightarrow \Lambda^k_j = \frac{1}{2}\,\mathrm{tr}\,\sigma^k s \check{\sigma}_j s^\star, \quad \Lambda \in \mathbf{SO}_0(1,3) \cong \mathbf{SL}(\mathbb{C}^2)/\{\pm \mathbf{1}_2\}$$

with Weyl matrices $\sigma^k = (\mathbf{1}_2, \vec{\sigma}) = \check{\sigma}_k$. In the Cartan representation, the spacetime translations are hermitian 2×2-matrices, that is representatives of the classes of all complex matrices $\log \mathbf{GL}(\mathbb{C}^2) \cong \mathbb{R}^4 \oplus (i\mathbb{R})^4$ with respect to the unitary ones $\log \mathbf{U}(2) \cong (i\mathbb{R})^4$

$$x = x^\star \Subset \log \mathbf{GL}(\mathbb{C}^2)/\log \mathbf{U}(2)$$

[8] The funny double element symbol means a representative of a coset, that is $g \Subset G/H \iff g \in gH \in G/H$.

Global spacetime arises by exponentiation and is given by the classes of the full group $\mathbf{GL}(\mathbb{C}^2)$ with respect to the unitary phases $\mathbf{U}(2)$, the moduli of $\mathbf{GL}(\mathbb{C}^2)$

$$\text{spacetime: } \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \cong \mathbf{D}(2) = \exp \mathbb{R}^4 \cong \mathbb{R}^4$$

The causal structure of spacetime is the spectral order (Rickart, 1960) of the $C*$-algebra $\log \mathbf{GL}(\mathbb{C}^2)$.

The noncompact symmetric space $\mathbf{D}(2)$ has, analogue to its compact counterpart $\mathbf{U}(2)$ with $\mathbf{U}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2)$, a product decomposition into Abelian causal time group $\mathbf{D}(\mathbf{1}_2)$ and real three-dimensional position (boost) manifold $\mathbf{SD}(2)$

$$\mathbf{D}(2) = \mathbf{D}(1_2) \times \mathbf{SD}(2), \quad \mathbf{SD}(2) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$$

Both symmetric spaces have real rank 2—also indicated in the notation $\mathbf{U}(2)$ and $\mathbf{D}(2)$—which reflects both the number of independent invariants and the dimension of a maximal Abelian Cartan subgroup (flat submanifold; Helgason, 1978), arising as factor of the two-sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$ in the polar decomposition

$$\begin{aligned}
\mathbf{U}(2) &= \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2) \cong \mathbf{U}(1) \circ \mathbf{SO}(2) \times \mathbf{SO}(3)/\mathbf{SO}(2) \\
\mathbf{D}(2) &= \mathbf{D}(\mathbf{1}_2) \times \mathbf{SD}(2) \cong \mathbf{D}(1) \times \mathbf{SO}_0(1,1) \times \mathbf{SO}(3)/\mathbf{SO}(2)
\end{aligned}$$

For the decomposition of the real four-dimensional tangent spaces (Lie algebra for $\mathbf{U}(2)$) with the Lie algebra of the Cartan subgroup, the sphere factor remains unchanged

$$\begin{aligned}
\log \mathbf{U}(2) &= \log \mathbf{U}(\mathbf{1}_2) \oplus \log \mathbf{SU}(2) \\
&\cong \log \mathbf{U}(1) \ \ \oplus [\log \mathbf{SO}(2) \times \mathbf{SO}(3)/\mathbf{SO}(2)] \\
\log \mathbf{D}(2) &= \log \mathbf{D}(\mathbf{1}_2) \oplus \log \mathbf{SD}(2) \\
&\cong \log \mathbf{D}(1) \ \ \oplus [\log \mathbf{SO}_0(1,1) \times \mathbf{SO}(3)/\mathbf{SO}(2)]
\end{aligned}$$

The representations of noncompact spacetime $\mathbf{D}(2)$ and compact internal group $\mathbf{U}(2)$ are characterized by two invariants from a continuous spectrum for a Cartan subgroup $\mathbf{D}(1) \times \mathbf{SO}_0(1,1)$) and from a discrete spectrum for a Cartan subgroup $\mathbf{U}(1) \circ \mathbf{SO}(2)$ respectively. Minkowski spacetime $\mathbb{R}^4$ in the Cartan representation by $\mathbf{U}(2)$-hermitian 2×2-matrices has the familiar conjugate adjoint $\mathbf{GL}(\mathbb{C}^2)$-transformation behavior to be compared with the adjoint action of the compact group $\mathbf{U}(2)$ on its Lie algebra $\log \mathbf{U}(2) \cong (i\mathbb{R})^4$

$$\begin{aligned}
&g \in \mathbf{GL}(\mathbb{C}^2), \quad x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \in \log \mathbf{D}(2) \quad \Rightarrow x \mapsto g \circ x \circ g^* \\
&u \in \mathbf{U}(2), \quad i\alpha = i \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \in \log \mathbf{U}(2) \Rightarrow i\alpha \mapsto u \circ i\alpha \circ u^* \\
&\qquad u^* = u^{-1}
\end{aligned}$$

However, in contrast to the decomposition of the $\mathbf{U}(2)$-Lie algebra into Abelian $\mathbf{U}(\mathbf{1}_2)$ and simple $\mathbf{SU}(2)$-contribution, compatible with the adjoint $\mathbf{U}(2)$-action, the decomposition of spacetime $\mathbf{D}(2)$ and its tangent space into time and position is not compatible with the action of the Lorentz group

$$\begin{array}{lll} u \in \mathbf{U}(2), & \log \mathbf{U}(2) \ni i\alpha = i\alpha_0 \mathbf{1}_2 + i\vec{\alpha}\vec{\sigma}, & \begin{cases} u \circ i\alpha_0 \mathbf{1}_2 \circ u^* \in \log \mathbf{U}(\mathbf{1}_2) \\ u \circ i\vec{\alpha}\vec{\sigma} \circ u^* \in \log \mathbf{SU}(2) \end{cases} \\ s \in \mathbf{SL}(\mathbb{C}^2), & \log \mathbf{D}(2) \ni x = x_0 \mathbf{1}_2 + \vec{x}\vec{\sigma} & \\ & \text{in general} & \begin{cases} s \circ x_0 \mathbf{1}_2 \circ s^* \notin \log \mathbf{D}(\mathbf{1}_2) \\ s \circ \vec{x}\vec{\sigma} \circ s^* \notin \log \mathbf{SD}(2) \end{cases} \end{array}$$

Both symmetric spaces are parametrizable by exponentiating the tangent space, for example, in the polar Cartan decomposition

$$\begin{aligned} \log \mathbf{U}(2) \ni i\alpha &= u\left(\frac{\vec{\alpha}}{|\vec{\alpha}|}\right) \circ i(\alpha_0 \mathbf{1}_2 + |\vec{\alpha}|\sigma_3) \circ u^*\left(\frac{\vec{\alpha}}{|\vec{\alpha}|}\right) \\ \Rightarrow \exp i\alpha &= u\left(\frac{\vec{\alpha}}{|\vec{\alpha}|}\right) \circ e^{i(\alpha_0 \mathbf{1}_2 + |\vec{\alpha}|\sigma_3)} \circ u^*\left(\frac{\vec{\alpha}}{|\vec{\alpha}|}\right) \in \mathbf{U}(2) \\ \log \mathbf{D}(2) \ni x &= u\left(\frac{\vec{x}}{r}\right) \circ (x_0 \mathbf{1}_2 + r\sigma_3) \circ u^*\left(\frac{\vec{x}}{r}\right), \quad r = |\vec{x}| \\ \Rightarrow \exp x &= u\left(\frac{\vec{x}}{r}\right) \circ e^{x_0 \mathbf{1}_2 + r\sigma_3} \circ u^*\left(\frac{\vec{x}}{r}\right) \in \mathbf{D}(2) \end{aligned}$$

The diagonalization of $\mathbf{D}(2)$ and $\mathbf{U}(2)$ with the sphere operations

$$u\left(\frac{\vec{x}}{r}\right) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \Subset \mathbf{SU}(2)/\mathbf{U}(1) \cong \mathbf{SO}(3)/\mathbf{SO}(2)$$

defines $\{i\alpha_0, i|\vec{\alpha}|\}$ as Cartan coordinates for the internal group and $\{x_0, r\}$ (time and radial translations) as Cartan coordinates for spacetime.

Similar to the local-global group isomorphism for time $\mathbb{R} \cong \exp \mathbb{R} = \mathbf{D}(1)$ one has the manifold isomorphy for spacetime $\mathbb{R}^4 \cong \exp \mathbb{R}^4 = \mathbf{D}(2)$. Via their embedment as future cones, $\mathbf{D}(1)$ and $\mathbf{D}(2)$ are parametrizable with tangent space $\mathbb{R}$ and $\mathbb{R}^4$ coordinates

$$\begin{array}{llll} t \in \mathbb{R} & \Rightarrow \mathbf{D}(1) \ni e^t = \epsilon(s)s & \in \mathbb{R}^+ & \text{with } s \in \mathbb{R}, \quad s^2 = e^{2t} \\ x \in \mathbb{R}^4 & \Rightarrow \mathbf{D}(2) \ni e^x = \epsilon(y_0)\vartheta(y^2)y & \in (\mathbb{R}^4)^+ & \text{with } y \in \mathbb{R}^4, \begin{cases} y_0^2 = e^{2x_0} \\ |\vec{y}| = e^r \end{cases} \end{array}$$

5.4. Time in Spacetime

A dynamics in quantum mechanics arises from representations of the time group $\mathbf{D}(1) \cong \exp \mathbb{R}$ whose representation spaces are realized in the Schrödinger

Table I. Embedding Time into Spacetime

	Time $\mathbf{D}(1) = \mathbf{GL}(\mathbb{C})/\mathbf{U}(1)$	$\hookrightarrow$	Spacetime $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$
Quantum theory	Quantum mechanics	$\hookrightarrow$	Quantum fields
Cartan subgroup	$\mathbf{D}(1)$	$\hookrightarrow$	$\mathbf{D}(1) \times \mathbf{SO}_0(1, 1)$
Full group	$\mathbf{GL}(\mathbb{C})$	$\hookrightarrow$	$\mathbf{GL}(\mathbb{C}^2)$
Tangent space (translations)	$\mathbb{R}$	$\hookrightarrow$	$\mathbb{R}^4$
Future	$t = \epsilon(t)t$	$\hookrightarrow$	$x = \epsilon(x_0)\vartheta(x^2)x$
	$\mathbb{R}^+ \cong \mathbf{D}(1)$		$(\mathbb{R}^4)^+ \cong \mathbf{D}(2)$
Particles (states)	$\mathbf{D}(1) \to \mathbf{U}(1)$	$\cong$	$\mathbf{D}(1) \to \mathbf{U}(1)$
Interactions	Not intrinsic		$\mathbf{SO}_0(1, 1) \to \mathbf{U}(1, 1)$

picture by wave functions depending on position translations. The quantum mechanical relevant time structure is a proper substructure of spacetime, modeled by the homogeneous space $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ and represented by quantum fields. The quantum mechanical energy eigenstates for compact $\mathbf{D}(1)$-representations are embedded as spacetime particles. The strict future cone with dimension four in flat spacetime being isomorphic to nonlinear spacetime $\mathbf{D}(2)$ contains not only the totally ordered one-dimensional causal subgroup $\mathbf{D}(1)$, it leaves room for a three-dimensional position submanifold $\mathbf{SD}(2)$ whose noncompact dilatations $\mathbf{SO}_0(1, 1)$ characterize spacetime interactions. The particle contributions, unitarily representing $\mathbf{D}(1)$, have to be supplemented in relativistic quantum theories by nonparticle ones to implement genuine $\mathbf{SO}_0(1, 1)$-representations. The nonparticle contributions are a genuine intrinsic feature of spacetime $\mathbf{D}(2)$ without analogue in quantum mechanics. There the interactions, such as the Coulomb potential for atoms, have to be put in by hand (see Table I).

6. TWO CONTINUOUS INVARIANTS FOR SPACETIME REPRESENTATIONS

Since Yukawa, the unification of a causal time development (characterized by a particle mass $m_0 \geq 0$) with a position interaction (characterized by a range $\frac{1}{m_3}$, $m_3 \geq 0$) in one spacetime Klein–Gordon equation for an $\epsilon(x_0)$-multiplied quantization distribution with one mass $m \geq 0$

$$\left.\begin{array}{l} \left(\frac{d^2}{dt^2} + m_0^2\right) \frac{e^{i|t|m_0}}{im_0} = 2\delta(t) \\ \left(-\frac{\partial^2}{\partial \vec{x}^2} + m_3^2\right) \frac{e^{-rm_3}}{2\pi r} = 2\delta(\vec{x}) \end{array}\right\} \hookrightarrow (\partial^2 + m^2)\epsilon(x_0)\frac{\mathbf{s}(x \mid m)}{m}$$

$$= 2\delta(x) \quad \text{with } m_0 = m_3 = m$$

seems to be an obvious relativistic bonus—all interactions can be interpreted as particle induced.

Particle fields with a Dirac energy–momentum distribution in their quantization

$$i\mathbf{s}(x \mid m) = \int \frac{d^4q}{(2\pi)^3} \epsilon(q_0) m \delta(q^2 - m^2)\, e^{ixq}$$

give by position-integration-representation matrix elements of the Abelian time group $\mathbf{D}(1) \cong \exp \mathbb{R}$ in $\mathbf{SO}(2)$

$$\begin{aligned} &\mathbf{D}(1) \to \mathbb{C} \\ &e^{x_0} \mapsto \int d^3x\, i\mathbf{s}(x \mid m) = \int dE\, m\epsilon(E)\delta(E^2 - m^2)\, e^{ix_0E} = i\ \sin x_0 m \end{aligned}$$

The appropriate distribution for a representation of the position symmetric space $\mathbf{SD}(2) \cong \exp \mathbb{R}^3$ arises from a derived energy–momentum Dirac distribution

$$\frac{i\mathbf{s}^{\text{dip}}(x \mid m)}{m} = -\frac{d}{dm^2} \frac{i\mathbf{s}(x \mid m)}{m} = \int \frac{d^4q}{(2\pi)^3} \epsilon(q_0)\delta'(q^2 - m^2)\, e^{ixq}$$

Time integration leads to a Dirac distribution for the invariant and to $\mathbf{SD}(2)$-representation matrix elements in $\mathbf{SO}_0(1, 1)$

$$\begin{aligned} &\mathbf{SD}(2) \to \mathbb{C} \\ &e^{-\vec{x}} \mapsto 4\pi \int dx_0 \epsilon(x_0) \mathbf{s}^{\text{dip}}(x \mid m) = \int dQ\, m\delta(Q^2 - m^2)\, e^{-r|Q|} = e^{-rm} \end{aligned}$$

The Dirac energy–momentum distribution for time with characterizing second order differential equation in contrast to the derived distribution for position with characterizing fourth order differential equation

$$\left(\frac{d^2}{dt^2} + m^2\right)\frac{e^{i|t|m}}{im} = 2\delta(t),\ \left(-\frac{\partial^2}{\partial \vec{x}^2} + m^2\right)^2 \frac{e^{-rm}}{4\pi m} = 2\delta(\vec{x})$$

reflect the different dimensions 1 and 3 of the time group $\mathbf{D}(1)$ and the position manifold $\mathbf{SD}(2)$ respectively.

The association of energy–momentum singularities to representation invariants for $\mathbf{D}(1)$ (time) and $\mathbf{SD}(2)$ (position) respectively is blurred since a decomposition of the spacetime tangent Minkowski translations $\mathbb{R}^4 \ni x = \mathbf{1}_2 x_0 + \vec{\sigma}\vec{x}$ into time and position translations is not compatible with the action of the Lorentz group $\mathbf{SO}_0(1, 3)$. The Dirac distribution has also a nontrivial projection for the position $\mathbf{SD}(2)$ structure

$$2\pi \int dx_0 \epsilon(x_0) \mathbf{s}(x \mid m) = m\ \frac{e^{-rm}}{r}$$

and the derived Dirac distribution a nontrivial projection for time $\mathbf{D}(1)$ representations

$$\int d^3x\ i\mathbf{s}^{\mathrm{dip}}(x \mid m) = i\ \frac{\sin x_0 m - x_0 m\ \cos x_0 m}{2m^2}$$

The position projection of the Dirac distribution leads to a Yukawa force which is not a matrix element of an $\mathbf{SD}(2)$-representation, but only of its tangent position translations $\mathbb{R}^3$. The time projection of the derived Dirac distribution leads to matrix elements of reducible nondecomposable $\mathbf{D}(1)$-representations.

Related to the two Cartan coordinates $\{x_0, r\}$ that reflect the rank 2 of the noncompact homogeneous manifold $\mathbf{D}(2)$, that is two Abelian subgroups $\mathbf{D}(\mathbf{1}_2)$ (time) and $\mathbf{SO}_0(1, 1)$ as a dilatation subgroup of the position manifold $\mathbf{SD}(2)$, two invariants $\{m_0^2, m_3^2\}$ have to characterize the $\mathbf{D}(2)$-representations. The definite unitary representations $\mathbf{D}(\mathbf{1}_2) \ni e^{x_0 \mathbf{1}_2} \mapsto e^{\pm x_0 i m_0} \in \mathbf{U}(1)$ are characterized by a particle mass m_0^2. A second mass m_3^2 characterizes the indefinite unitary representation $\mathbf{SO}_0(1, 1) \ni e^{\pm r} \mapsto e^{\pm r m_3} \in \mathbf{SU}(1, 1)$ with an interaction range $\frac{1}{m_3}$ and without particle asymptotics. There is no group theoretical reason to identify both scales $m_0^2 = m_3^2$; in general, the representations of spacetime $\mathbf{D}(2)$ come with two different scales whose ratio $\frac{m_3^2}{m_0^2}$ is a representation characteristic of a physically important constant. The ratio of the characterizing invariants for particle and interaction should be seen in analogy with the relative normalization of time and position translations $\begin{pmatrix} \tau^2 & 0 \\ 0 & -\ell^2\mathbf{1}_3 \end{pmatrix}$ as given with the speed of light $c^2 = \frac{\ell^2}{\tau^2}$.

7. RESIDUAL REPRESENTATIONS

Before the definition of residual representations in general their structure will be exemplified in the familiar example of the compact and noncompact abelian groups $\mathbf{U}(1)$ and $\mathbf{D}(1)$.

7.1. Residual U(1) × D(1)-Representations

An irreducible representation of the complex Abelian group $\exp \mathbb{C}$ can be written as residue of its eigenvalue by using the complex Lie algebra forms $Q \in \mathbb{C}$

$$\exp \mathbb{C} \ni e^z \mapsto e^{z\zeta} = \oint \frac{dQ}{2i\pi} \frac{1}{Q - \zeta}\, e^{zQ},\ \zeta \in \mathbf{irrep} \exp \mathbb{C} \cong \mathbb{C}$$

which, with the canonical conjugation, gives for the irreducible $\mathbf{U}(1)$ and $\mathbf{D}(1)$-representations, necessarily in $\mathbf{U}(1)$

$$\mathbf{U}(1) \ni e^{i\alpha} \mapsto e^{i\alpha Z} = \int dq \delta(q - Z)\, e^{i\alpha q} = \oint \frac{dq}{2i\pi} \frac{1}{q - Z}\, e^{i\alpha q} \in \mathbf{U}(1)$$
$$Z \in \mathbf{irrep}\, \mathbf{U}(1) \cong \mathbb{Z}$$

$$\mathbf{D}(1) \ni e^t \mapsto e^{tim} = \int dq\delta(q-m)\,e^{itq} = \oint \frac{dq}{2i\pi}\frac{1}{q-m}\,e^{itq} \in \mathbf{U}(1)$$
$$im \in \mathbf{irrep}\,\mathbf{D}(1) \cong i\mathbb{R}$$

with the neutral representations for $Z = 0$ and $m = 0$ respectively. The integrations for the compact and noncompact group are related to each other via the Lie algebras and their forms by multiplication with the imaginary unit i

$$\text{for compact } \mathbf{U}(1)\ (i\alpha, q) \leftrightarrow (t, iq) \text{ for noncompact } \mathbf{D}(1)$$

Measures of the integer winding numbers Z as invariants of the compact group $\mathbf{U}(1)$ lead to Fourier series as measured $\mathbf{U}(1)$-representations

$$\mu : \mathbf{irrep}\,\mathbf{U}(1) \to \mathbb{R}, \quad Z \mapsto \mu(Z)$$
$$\mathbf{meas\ irrep}\,\mathbf{U}(1) \ni \mu \mapsto D^\mu \in \mathbf{rep}\,\mathbf{U}(1)$$
$$\mathbf{U}(1) \ni e^{i\alpha} \mapsto D^\mu(\alpha) = \sum_{Z\in\mathbb{Z}} \mu(Z)\,e^{i\alpha Z}$$

The continuous irreducible representation classes for $\mathbf{D}(1)$ characterized by imaginary numbers im have Lebesque measure dm based real valued measures giving rise to Fourier integrals as measured $\mathbf{D}(1)$-representations

$$\mu : \mathbf{irrep}\,\mathbf{D}(1) \to \mathbb{R}, \quad m \mapsto \mu(m)$$
$$\mathbf{meas\ irrep}\,\mathbf{D}(1) \ni \mu \mapsto D^\mu \in \mathbf{rep}\,\mathbf{D}(1)$$
$$\mathbf{D}(1) \ni e^t \mapsto D^\mu(t) = \int dm\,\mu(m)\,e^{itm}$$

where also matrix elements of reducible nondecomposable representations may occur by using derivatives with respect to the invariant

$$\mu(m) = \sum_{N=0,1,\ldots} \mu_N(m)\left(\frac{d}{dm}\right)^N$$

7.2. The Definition of Residual Representations

Residual representations are complex functions on a real finite dimensional symmetric space G, for example, a Lie group, with tangent space (Lie algebra) $\log G \cong \mathbb{R}^n$, as given previously for $\mathbf{U}(1)$ and $\mathbf{D}(1)$ and in the following for $\mathbf{SU}(2)$ and $\mathbf{SL}(\mathbb{C}^2)$ and generalized to the position manifold $\mathbf{SD}(2)$ and the spacetime manifold $\mathbf{D}(2)$.

The equivalence classes **irrep** G of the irreducible G-representations are characterizable by invariants, taken from a rational spectrum for a compact and also from a continuous spectrum for a noncompact Cartan subgroup. The weights (eigenvalues) for the symmetric space G are a submodule of the linear forms[9] $q \in (\log G)^T$ of the tangent space $x \in \log G$. The invariants $\{I_1, \ldots, I_r\}$, characterizing an irreducible representation, are related to multilinear tangent space forms (monomials in the weights). Appropriate measures $d^n q\, I(q)$ of the linear forms, which can be written with a Lebesque measure basis and a distribution of the tangent space forms $(\log G)^T \cong \mathbb{R}^n$ lead to matrix elements of irreducible symmetric space representations

$$\begin{aligned} I : \mathbb{R}^n &\to \mathbb{R}, & q &\mapsto I(q) \\ D : \mathbf{meas}\,\mathbb{R}^n &\to \mathbf{irrep}\, G, & I &\mapsto D^I \\ D^I : G &\to \mathbb{C}, & g(x) &\mapsto D^I(x) = \int d^n q\, I(q)\, e^{ixq} \end{aligned}$$

The complex generalized functions $I(q)$ have poles at the values for the invariants characterizing an irreducible representation, the distributions come as quotients of two polynomials $I(q) = \frac{P_N(q)}{P_D(q)}$. D^I is called a residual representation of G with $I(q)$ a residual group distribution.

Measured representations for a symmetric space (Lie group) G integrate irreducible G-representations with a measure $d^r I\, \mu(I)$ of the invariants

$$\begin{aligned} \mu : \mathbf{irrep}\, G &\to \mathbb{R}, & I &\mapsto \mu(I) \\ D : \mathbf{meas\ irrep}\, G &\to \mathbf{rep}\, G, & \mu &\mapsto D^\mu \\ D^\mu : G &\to \mathbb{C}, & g(x) &\mapsto D^\mu(x) = \int d^r I\, \mu(I) D^I(x) \end{aligned}$$

The product in the algebra of the representation classes $\mathbf{rep}\, G$ is implemented via the convolution of the distributions for the matrix elements of the product representation

$$D^{I_1} \otimes D^{I_2} = D^{I_1 * I_2}$$

In the following, these general structures will be concretized for the groups and symmetric spaces relevant for the spacetime model $\mathbf{D}(2)$.

7.3. Residual SO(2) × SO₀(1, 1)-Representations

The real Abelian group $\mathbf{SO}(2) \times \mathbf{SO}_0(1,1)$ has its irreducible self-dual complex representations in the two types of two-dimensional unitary groups, the definite unitary $\mathbf{SU}(2)$ or the indefinite unitary $\mathbf{SU}(1,1)$

$$\mathbf{SO}(2) \times \mathbf{SO}_0(1,1) \to \begin{cases} \mathbf{SO}(2) & \subset \mathbf{SU}(2) \\ \mathbf{SO}_0(1,1) & \subset \mathbf{SU}(1,1) \end{cases}$$
$$e^{(i\alpha + x)\sigma^3} \mapsto e^{(i\alpha Z + x\delta)\sigma^3}$$

[9] The linear forms (dual space) of a vector space V are denoted by V^T.

The unitary groups $\mathbf{SU}(2)$ and $\mathbf{SU}(1,1)$ define the weights (Z,δ) of the principal (compact) and supplementary (noncompact) representations respectively.

The principal $\mathbf{SO}(2)\times\mathbf{SO}_0(1,1)$-weights coincide with the $\mathbf{U}(1)\times\mathbf{D}(1)$-weights $\mathbb{Z}\times i\mathbb{R}$. An integer eigenvalue pair $\{\pm Z\}$ characterizes a self-dual $\mathbf{SO}(2)$-representation

$$\mathbf{SO}(2)\ni\begin{pmatrix}\cos\alpha & i\,\sin\alpha\\ i\,\sin\alpha & \cos\alpha\end{pmatrix}\mapsto\begin{pmatrix}\cos\alpha Z & i\,\sin\alpha Z\\ i\,\sin\alpha Z & \cos\alpha Z\end{pmatrix}$$
$$\cong\begin{pmatrix}e^{i\alpha Z} & 0\\ 0 & e^{-i\alpha Z}\end{pmatrix}\in\mathbf{SU}(2)$$

leading to a quadratic natural number valued invariant Z^2. An imaginary continuous eigenvalue pair $\{\pm im\}$ characterizes a self-dual compact $\mathbf{SO}_0(1,1)$-representation

$$\mathbf{SO}_0(1,1)\ni\begin{pmatrix}\cosh x & \sinh x\\ \sinh x & \cosh x\end{pmatrix}\mapsto\begin{pmatrix}\cos xm & i\,\sin xm\\ i\,\sin xm & \cos xm\end{pmatrix}$$
$$\cong\begin{pmatrix}e^{ixm} & 0\\ 0 & e^{-ixm}\end{pmatrix}\in\mathbf{SU}(2)$$

with a continuous positive invariant $m^2\geq 0$

$$\mathbf{weights}\,\mathbf{SO}(2)=\{Z\}\cong\mathbb{Z},\qquad \mathbf{irrep}\,\mathbf{SO}(2)=\{|Z|\}\cong\mathbb{N}_0$$
$$\mathbf{weights}^{(2,0)}\mathbf{SO}_0(1,1)=\{im\}\cong i\mathbb{R},\qquad \mathbf{irrep}^{(2,0)}\mathbf{SO}_0(1,1)=\{m^2\}\cong\mathbb{R}^+$$

The new real $\mathbf{SO}_0(1,1)$-weights $m\in\mathbb{R}$ (supplementary) in contrast to the imaginary principal weights $im\in i\mathbb{R}$ given earlier come for dimensions $n\geq 2$ with the possibility of indefinite unitary groups. A supplementary $\mathbf{SO}_0(1,1)$-representation is characterized by a real continuous eigenvalue pair $\{\pm m\}$

$$\mathbf{SO}_0(1,1)\ni\begin{pmatrix}\cosh x & \sinh x\\ \sinh x & \cosh x\end{pmatrix}\mapsto\begin{pmatrix}\cosh xm & \sinh xm\\ \sinh xm & \cosh xm\end{pmatrix}$$
$$\cong\begin{pmatrix}e^{xm} & 0\\ 0 & e^{-xm}\end{pmatrix}\in\mathbf{SU}(1,1)$$

with a continuous negative definite invariant

$$\mathbf{weights}^{(1,1)}\mathbf{SO}_0(1,1)=\{m\}=\mathbb{R},\qquad \mathbf{irrep}^{(1,1)}\mathbf{SO}_0(1,1)=\{-m^2\}\cong\mathbb{R}^-$$

Residual representations in $\mathbf{SO}(2)$ (principal) with invariants $m^2\in\mathbb{R}^+$ can be formulated by distributions with the q-integration deformed as prescribed by $q^2\mp io$, which for an undeformed integration gives singularities at $m^2 \pm io = (|m| \pm io)^2$

$$e^{\pm i|tm|} = \int d^1q\,[m^2]^0_\pm(q)\,e^{itq}, \quad [m^2]^0_\pm(q) = \pm\frac{1}{i\pi}\frac{|m|}{q^2 \mp io - m^2}$$

$$\epsilon(t)e^{\pm i|tm|} = \int d^1q\,[m^2]^1_\pm(q)\,e^{itq}, \quad [m^2]^1_\pm(q) = \frac{1}{i\pi}\frac{q}{q^2 \mp io - m^2}$$

$$\text{for } \mathbf{SO}(2), \mathbf{SO}_0(1,1) \to \mathbf{SU}(2),\ m \in (\mathbb{Z}, \mathbb{R})$$

Residual representations in $\mathbf{SU}(1, 1)$ (supplementary) with invariants $-m^2 \in \mathbb{R}^-$ are obtained from residual representations in $\mathbf{SU}(2)$ (principal) by the real–imaginary exchange $(it, q) \leftrightarrow (x, iq)$

$$e^{-|xm|} = \int d^1q\,[-m^2]^0(q)\,e^{-ixq}, \quad [-m^2]^0(q) = \frac{1}{\pi}\frac{|m|}{q^2 + m^2}$$

$$-\epsilon(x)e^{-|xm|} = \int d^1q\,[-m^2]^1(q)\,e^{-ixq}, \quad [-m^2]^1(q) = \frac{1}{i\pi}\frac{q}{q^2 + m^2}$$

$$\text{for } \mathbf{SO}_0(1,1) \to \mathbf{SU}(1,1), \quad m \in \mathbb{R}$$

In the transition from the compact to the noncompact representation structure the invariant $\pm i|m|$ has to be replaced by $-|m|$

$$\text{for } \mathbf{SO}(2) \subset \mathbf{SU}(2)\ \pm\ i|m| \leftrightarrow -|m| \text{ for } \mathbf{SO}_0(1,1) \subset \mathbf{SD}(2)$$

The matrix elements for the representations in $\mathbf{SO}(2)$ and $\mathbf{SO}_0(1, 1)$ fulfill the second order differential equations

$$\left(\frac{d^2}{dt^2} + m^2\right)e^{\pm i|tm|} = \pm 2i|m|\delta(t), \qquad \left(\frac{d^2}{dx^2} - m^2\right)e^{-|xm|} = -2|m|\delta(x)$$

The product representations arise by convolution—for $\mathbf{SO}(2)$ with equal type, either $+io$ or $-io$—with the supindices $\{1, 0\}$ adding up modulo 2, for example,

$$\left.\begin{aligned} \left[m_1^2\right]^1_\pm * \left[m_2^2\right]^1_\pm &= \left[m_+^2\right]^0_\pm \\ \left[-m_1^2\right]^1 * \left[-m_2^2\right]^1 &= \left[-m_+^2\right]^0 \end{aligned}\right\} |m_+| = |m_1| + |m_2|$$

With the convolution the distributions

$$\mathbf{irrep}\,\mathbf{SO}(2) = \left\{q \mapsto [Z^2]^1_\pm(q) = \frac{1}{i\pi}\frac{q}{q^2 \mp io - Z^2}\ \middle|\ Z \in \mathbb{Z}\right\}$$

$$\mathbf{irrep}^{(2,0)}\mathbf{SO}_0(1,1) = \left\{q \mapsto [m^2]^1_\pm(q) = \frac{1}{i\pi}\frac{q}{q^2 \mp io - m^2}\ \middle|\ m \in \mathbb{R}\right\}$$

$$\mathbf{irrep}^{(1,1)}\mathbf{SO}_0(1,1) = \left\{q \mapsto [-m^2]^1(q) = \frac{1}{i\pi}\frac{q}{q^2 + m^2}\ \middle|\ m \in \mathbb{R}\right\}$$

generate the compact and noncompact self-dual Abelian representations respectively. The neutral representations arise for trivial invariant.

7.4. Residual Representations for Spin SU(2)

If the compact group **SO**(2) comes as Cartan subgroup in the special group $e^{-i\vec{x}\vec{\sigma}} \in \mathbf{SU}(2)$ with the Cartan polar decomposition

$$\mathbf{SU}(2) \cong \mathbf{SO}(2) \times \mathbf{SO}(3)/\mathbf{SO}(2)$$

residual representations employ the forms $\vec{q} \in \mathbb{R}^3$ of the tangent Lie algebra $\log \mathbf{SU}(2)$ (angular momenta) with the singularities of the distributions determined by the values of the invariant bilinear Killing form $\vec{q}^2$ as singularity location of a dipole

$$\text{for } \mathbf{SO}(2) \subset \mathbf{SU}(2):\; e^{\pm ir|m|} = \int d^3q[0, m^2]_\pm(\vec{q})\, e^{-i\vec{x}\vec{q}}$$

$$[0, m^2]_\pm(\vec{q}) = \pm\frac{1}{i\pi^2}\frac{|m|}{(\vec{q}^2 \mp io - m^2)^2}, \qquad m \in \mathbb{R}$$

This scalar representation and similar integrals can be obtained by derivations with respect to the invariant m^2 and the Lie parameter $\vec{x}$ from the in- and outgoing spherical waves

$$\int \frac{d^3q}{2\pi^2}\frac{1}{\vec{q}^2 \mp io - m^2}\, e^{-i\vec{x}\vec{q}} = \frac{e^{\pm ir|m|}}{r}, \quad m \in \mathbb{R}, \quad \vec{x} \neq 0$$

$$\frac{\partial}{\partial m^2} = \frac{1}{2|m|}\frac{\partial}{\partial |m|}, \qquad \frac{\partial}{\partial \vec{x}} = \frac{\vec{x}}{r}\frac{\partial}{\partial r}, \quad \left(\frac{\partial^2}{\partial \vec{x}^2} + m^2\right)\frac{e^{\pm ir|m|}}{r} = 4\pi\delta(\vec{x})$$

which, however, are no **SU**(2)-representation matrix elements because of the Lie parameter $\vec{x} = 0$ singularity.

The scalar matrix elements fulfill fourth order differential equations

$$\left(\frac{\partial^2}{\partial \vec{x}^2} + m^2\right)^2 e^{\pm ir|m|} = \mp 8\pi i|m|\delta(\vec{x})$$

Vector valued distributions represent nontrivially the two-sphere **SO**(3)/**SO**(2)

$$-\frac{\vec{x}}{r}\, e^{\pm ir|m|} = \int d^3q[1, m^2]_\pm(\vec{q})\, e^{-i\vec{x}\vec{q}}$$

$$= \int \frac{d^3q}{i\pi^2}\frac{\vec{q}}{(\vec{q}^2 \mp io - m^2)^2}\, e^{-i\vec{x}\vec{q}}, \quad m \in \mathbb{R}$$

leading to the matrix elements of the defining Pauli representation

$$\left.\begin{array}{l}\int d^3q\,[0,1]_\pm(\vec{q})\,e^{-i\vec{x}\vec{q}} = e^{\pm ir}\\ \int d^3q\,[1,1]_\pm(\vec{q})\,e^{-i\vec{x}\vec{q}} = -\frac{\vec{x}}{r}\,e^{\pm ir}\end{array}\right\} \leftrightarrow e^{-i\vec{x}\vec{\sigma}} = \mathbf{1}_2 \cos r - \frac{\vec{\sigma}\vec{x}}{r} i\ \sin r$$

The spherical dependence $\frac{\vec{x}}{r}$ replaces the $\epsilon(x)$-dependence for **SO**(2).

With the Lie algebra additive convolution product of the distributions for the irreducible residual **SU**(2)-representations

$$\textbf{irrep SU}(2) = \left\{\vec{q} \mapsto [1, m^2]_\pm(\vec{q}) = \frac{1}{i\pi^2}\frac{\vec{q}}{(\vec{q}^2 \mp io - m^2)^2}\ \middle|\ |m| = 2J \in \mathbb{N}_0\right\}$$

involving the neutral representation for trivial invariant $m = 0$ one can combine the matrix elements for all other representations, for example, the scalar ones with $|m_1| + |m_2| = |m_+|$

$$\frac{x^a}{r} e^{\pm ir|m_1|}\delta_{ab}\frac{x^b}{r}\ e^{\pm ir|m_2|} = e^{\pm ir|m_+|}$$

$$\begin{aligned}\left[1, m_1^2\right]_\pm \overset{j'=0}{*} \left[1, m_2^2\right]_\pm(\vec{q}) &= \left[0, m_+^2\right]_\pm(\vec{q})\\ &= \left(\frac{1}{i\pi^2}\right)^2 \int d^3q_1 d^3q_2 \frac{q_1^a}{\left(\vec{q}_1^2 \mp io - m_1^2\right)^2}\delta(\vec{q}_1 + \vec{q}_2 - \vec{q})\delta_{ab}\frac{q_2^a}{\left(\vec{q}_2^2 \mp io - m_2^2\right)^2}\\ &= \pm\frac{1}{i\pi^2}\frac{|m_+|}{\left(\vec{q}^2 \mp io - m_+^2\right)^2}\end{aligned}$$

or for the adjoint representation

$$\delta_{ab}\ \cos 2r + \frac{x_a x_b}{r^2}(1 - \cos 2r) + \epsilon_{abc}\frac{x_c}{r}\ \sin 2r$$

which arises for $|m_+| = |m_1| + |m_2| = 2$

$$\frac{x^a}{r}\ e^{\pm ir|m_1|}\frac{x^b}{r}\ e^{\pm ir|m_2|} = \frac{x^a x^b}{r^2}\ e^{\pm ir|m_+|}$$

$$\begin{aligned}\left[1, m_1^2\right]_\pm \overset{2J'=2}{*} \left[1, m_2^2\right]_\pm(\vec{q}) = \left(\frac{1}{i\pi^2}\right)^2 \int d^3q_1\ d^3q_2 \frac{q_1^a}{\left(\vec{q}_1^2 \mp io - m_1^2\right)^2}\\ \times\ \delta(\vec{q}_1 + \vec{q}_2 - \vec{q})\frac{q_2^a}{\left(\vec{q}_2^2 \mp io - m_2^2\right)^2}\end{aligned}$$

In general, the matrix elements of **SU**(2)-representations come as products of a homogeneous polynomial (spherical harmonics) of degree $2J'$ for the sphere

SO(3)/**SO**(2)-representation and an expontential for the Cartan subgroup **SO**(2) with winding numbers $\pm 2J$

$$\left\{ \left[\frac{\vec{x}}{r}\right]^{2J'} e^{\pm ir2J} \;\middle|\; 2J' \in \mathbb{N}_0,\, 2J \in \mathbb{N}_0 \right\}$$

$$\left[\frac{\vec{x}}{r}\right]^{0} = \{1\},\ \left[\frac{\vec{x}}{r}\right]^{1} = \left\{ \frac{x^a}{r} \;\middle|\; a = 1, 2, 3 \right\},\quad \left[\frac{\vec{x}}{r}\right]^{2} = \left\{ \frac{x^a x^b}{r^2} - \frac{\delta^{ab}}{3} \right\}, \ldots$$

Matrix elements of measured **SU**(2)-representations use real measures of the irreducible representations classes

$$\mu : \textbf{irrep SU}(2) \to \mathbb{R}, \qquad 2J \mapsto \mu(4J^2)$$
$$\textbf{meas irrep SU}(2) \ni \mu \mapsto D^{\mu}_{\pm} \in \textbf{rep SU}(2)$$

with the functions on the spin group **SU**(2)

$$\mathbf{SU}(2) \ni e^{i\vec{x}\vec{\sigma}} \mapsto D^{\mu}_{\pm}(\vec{x}) = \sum_{2j=0,1,\ldots} \mu(4J^2) \int \frac{d^3q}{i\pi^2} \frac{\vec{q}}{(\vec{q}^2 \mp io - 4J^2)^2}\, e^{-i\vec{x}\vec{q}}$$
$$= -\frac{\vec{x}}{r} \sum_{2j=0,1,\ldots} \mu(4J^2)\, e^{\pm ir2J}$$

7.5. Residual Representations for Position SD(2)

For the position manifold $e^{-\vec{x}\vec{\sigma}} \in \mathbf{SD}(2)$ with the Cartan polar decomposition

$$\mathbf{SD}(2) \cong \mathbf{SO}_0(1, 1) \times \mathbf{SO}(3)/\,\mathbf{SO}(2)$$

residual representations use the tangent space forms (momenta $\vec{q} \in \mathbb{R}^3$) and, in comparison to **SU**(2), the tangent space real–imaginary exchange for compact–noncompact

$$\text{for } \mathbf{SU}(2) \quad \left\{ \begin{array}{r} \text{Lie algebra and forms } (i\vec{x}, \vec{q}) \leftrightarrow (\vec{x}, i\vec{q}) \\ \text{invariant } \pm i|m| \leftrightarrow -|m| \end{array} \right\} \quad \text{for } \mathbf{SD}(2)$$

As for the Cartan subgroup $\mathbf{SO}_0(1, 1)$ there exists two types: The compact representations $\mathbf{SD}(2) \to \mathbf{SU}(2)$ (principal) with $\mathbf{SO}_0(1, 1) \to \mathbf{SO}(2)$ and the noncompact ones $\mathbf{SD}(2) \to \mathbf{SU}(1, 1)$ (supplementary) with faithful representations $\mathbf{SO}_0(1, 1) \to \mathbf{SO}_0(1, 1)$. Both are representations of the homogeneous position manifold in a unitary group, definite or indefinite.

From the Yukawa potential

$$\int \frac{d^3q}{2\pi^2} \frac{1}{\vec{q}^2 + m^2} e^{-\vec{x}i\vec{q}} = \frac{e^{-r|m|}}{r}, \quad m \in \mathbb{R}, \ \vec{x} \neq 0$$

$$\left(\frac{\partial^2}{\partial \vec{x}^2} - m^2 \right) \frac{e^{-r|m|}}{r} = -4\pi \delta(\vec{x})$$

which, by itself, is no **SD**(2)-representation matrix element because of the $\vec{x} = 0$ singularity, one obtains by derivations $\frac{\partial}{\partial m^2}$ and $\frac{\partial}{\partial \vec{x}}$ the scalar matrix elements, trivially representing the sphere **SO**(3)/**SO**(2)

$$\text{for } \mathbf{SO}_0(1,1) \subset \mathbf{SD}(2)\text{: } e^{-r|m|} = \int d^3q [0, -m^2](\vec{q})\, e^{-i\vec{x}\vec{q}}$$

$$[0, -m^2](\vec{q}) = \frac{1}{\pi^2} \frac{|m|}{(\vec{q}^2 + m^2)^2}, \quad m \in \mathbb{R}$$

and the fundamental noncompact residual **SD**(2)-representations using a vector valued distribution

$$-\frac{\vec{x}}{r} e^{-r|m|} = \int d^3q [1, -m^2](\vec{q})\, e^{-i\vec{x}\vec{q}} = \int \frac{d^3q}{i\pi^2} \frac{\vec{q}}{(\vec{q}^2 + m^2)^2} e^{i\vec{x}\vec{q}}, \quad m \in \mathbb{R}$$

This has to be compared with the elements in the defining representation

$$e^{-\vec{x}\vec{\sigma}} = \mathbf{1}_2 \cosh r - \frac{\vec{\sigma}\vec{x}}{r} \sinh r$$

The scalar matrix elements fulfill fourth order differential equations

$$\left(\frac{\partial^2}{\partial \vec{x}^2} - m^2 \right)^2 e^{-r|m|} = 8\pi |m| \delta(\vec{x})$$

$$\left(\frac{\partial^2}{\partial \vec{x}^2} + m^2 \right)^2 e^{\pm ir|m|} = \mp 8\pi i |m| \delta(\vec{x}), \quad m \in \mathbb{R}$$

In contrast to the spin group **SU**(2) where the representations of the compact Cartan subgroup **SO**(2) and the sphere **SO**(3)/**SO**(2) go both with discrete invariants $2J', 2J \in \mathbb{N}_0$, arising as degree of the spherical harmonics and as winding numbers, the continuous invariant $m^2 \in \mathbb{R}^+$ of the noncompact Cartan group $\mathbf{SO}_0(1,1)$-representation in the case of the position manifold **SD**(2) is taken from a different spectrum as the discrete invariant $2J' \in \mathbb{N}_0$ for the sphere **SO**(3)/**SO**(2)-representations. Again the convolution products of the distributions for the fundamental residual **SD**(2)-representations

$$\mathbf{irrep}^{(1,1)}\, \mathbf{SD}(2) = \left\{ \vec{q} \mapsto [1, -m^2](\vec{q}) = \frac{1}{i\pi^2} \frac{\vec{q}}{(\vec{q}^2 + m^2)^2} \middle| m \in \mathbb{R} \right\}$$

$$\mathbf{irrep}^{(2,0)}\, \mathbf{SD}(2) = \left\{ \vec{q} \mapsto [1, m^2]_\pm(\vec{q}) = \frac{1}{i\pi^2} \frac{\vec{q}}{(\vec{q}^2 \mp io - m^2)^2} \middle| m \in \mathbb{R} \right\}$$

define the matrix elements of the **SD**(2)-representations. The representations for trivial invariant $m = 0$ will be called neutral.

Measured **SD**(2)-representations use real measures of the continuous invariants

$$\mu : \textbf{irrep SD}(2) \to \mathbb{R}, \qquad m^2 \mapsto \mu(m^2)$$

$$\textbf{meas irrep SD}(2) \ni \mu \mapsto D^\mu \in \textbf{rep SD}(2)$$

with the functions on the position manifold **SD**(2)

$$\textbf{SD}(2) \ni e^{-\vec{x}\vec{\sigma}} \mapsto D^\mu(\vec{x})$$

$$= \begin{cases} \int_0^\infty dm^2 \mu(m^2) \int \frac{d^3q}{i\pi^2} \frac{\vec{q}}{(\vec{q}^2 + m^2)^2} \, e^{-i\vec{x}\vec{q}} \\ \quad = -\frac{\vec{x}}{r} \int_0^\infty dm^2 \mu(m^2) \, e^{-r|m|} \quad \text{for } \textbf{rep}^{(1,1)} \, \textbf{SD}(2) \\ \text{and} \\ \int_0^\infty dm^2 \mu(m^2) \int \frac{d^3q}{i\pi^2} \frac{\vec{q}}{(\vec{q}^2 \mp io - m^2)^2} \, e^{-i\vec{x}\vec{q}} \\ \quad = -\frac{\vec{x}}{r} \int_0^\infty dm^2 \mu(m^2) \, e^{\pm ir|m|} \quad \text{for } \textbf{rep}^{(2,0)} \, \textbf{SD}(2) \end{cases}$$

The two integrations in measured representation matrix elements go over the tangent space forms $\int d^3q$ and the invariants $\int_0^\infty dm^2$ with the dimensions 3 and 1 of the symmetric space **SD**(2) and a Cartan subgroup $\mathbf{SO}_0(1, 1)$ respectively. Matrix elements of reducible nondecomposable representations occur by using derivatives with respect to the invariant

$$\mu(m^2) = \sum_{N=0,1,\ldots} \mu_N(m^2) \left(\frac{d}{dm^2} \right)^N$$

8. RESIDUAL REPRESENTATIONS OF SPACETIME

Matrix elements of representations of a symmetric space (Lie group) can be formulated as residues for characterizing invariant singularities of their tangent translation (Lie algebra) forms. For the groups **U**(1), **D**(1), **SU**(2) and the position manifold **SD**(2), as done in the former sections, this is only a reformulation of known structures. Residual representations constitute a genuine formulation for the rank 2 symmetric spacetime **D**(2). Two values for the Lorentz invariant energy–momentum square q^2 characterize the action of the causal group **D**(1) and the position manifold **SD**(2).

Representations of spacetime

$$\mathbf{D}(2) = \mathbf{D}(\mathbf{1}_2) \times \mathbf{SD}(2) \cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$$
$$\cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}_0(1,1) \times \mathbf{SO}(3)/\mathbf{SO}(2)$$

will be formulated as Fourier transforms of energy–momentum distributions, compatible with the action of the Lorentz group $\mathbf{SO}_0(1,3)$ on the tangent Minkowski spacetime. The two invariant masses characterizing the representations are implemented via singularities.

The irreducible residual representation matrix elements of spacetime $\mathbf{D}(2)$, parametrizable with causal vectors $x\vartheta(x^2) \in \mathbb{R}^4$ in tangent Minkowski spacetime where the two reflected points $\{\pm x\vartheta(x^2)\}$ and, equally, their representation images have to be identified

$$\mathbf{D}(2) \ni x\vartheta(x^2) \mapsto \left(m_0^2; 1, -m_3^2\right)(x) = \int d^4q \left[m_0^2; 1, -m_3^2\right](q)\, e^{ixq}$$

involve a two factorial energy–momentum distribution

$$\textbf{irrep } \mathbf{D}(2)$$
$$= \left\{ q \mapsto \left[m_0^2; 1, -m_3^2\right](q) = \frac{1}{i\pi^3} \frac{2q}{\left(q_{\mathrm{P}}^2 - m_0^2\right)\left(q_{\mathrm{P}}^2 - m_3^2\right)^2} \middle| m_0, m_3 \in \mathbb{R} \right\}$$

It describes the Lorentz compatible embedding for the representation of the two $\mathbf{D}(2)$-factors and involves a simple pole (particle singularity) for the compact representation of a Cartan subgroup time

$$\text{for } \mathbf{D}(\mathbf{1}_2) \to \mathbf{SU}(2)\text{: } \frac{1}{q_{\mathrm{P}}^2 - m_0^2}$$
$$\text{with pole location for } \vec{q} = 0\text{: } q_0^2 = m_0^2$$

and a dipole (interaction singularity) for the noncompact representation of the position symmetric space $\mathbf{SD}(2)$ with Cartan subgroup $\mathbf{SO}_0(1,1)$

$$\text{for } \mathbf{SD}(2) \supset \mathbf{SO}_0(1,1) \to \mathbf{SU}(1,1)\text{: } \frac{1}{\left(q_{\mathrm{P}}^2 - m_3^2\right)^2}$$
$$\text{with dipole location for } q_0 = 0\text{: } \vec{q}^{\,2} = -m_3^2$$

The two-sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$ nontrivially representing factor is given by $\{q^j\}_{j=0}^3$ in the numerator.

The Fourier transform of a principal value distribution is the causal Fourier transform of a Dirac distribution and vice versa

$$\int \frac{d^4q}{i\pi} \frac{1}{q_{\mathrm{P}}^2 - m^2} \begin{pmatrix} e^{ixq} \\ \epsilon(x_0)\, e^{ixq} \end{pmatrix} = \int d^4q\ \epsilon(q_0)\delta(q^2 - m^2) \begin{pmatrix} \epsilon(x_0)\, e^{ixq} \\ e^{ixq} \end{pmatrix}$$

The matrix elements of the measured spacetime representations as **D**(2)-functions involve a measure for the two continuous invariants

$$\begin{aligned}
&\mu : \textbf{irrep}\ \mathbf{D}(2) \to \mathbb{R}, \qquad \left(m_0^2, m_3^2\right) \mapsto \mu\left(m_0^2, m_3^2\right)\\
&\textbf{meas irrep}\ \mathbf{D}(2) \ni \mu \mapsto D^{\mu} \ni \textbf{rep}\ \mathbf{D}(2)\\
&\mathbf{D}(2) \ni x\vartheta(x^2) \mapsto D^{\mu}(x)\\
&= \int_0^{\infty} dm_0^2\, dm_3^2\, \mu\left(m_0^2, m_3^2\right) \int \frac{d^4q}{i\pi^3} \frac{2q}{\left(q_{\mathrm{P}}^2 - m_0^2\right)\left(q_{\mathrm{P}}^2 - m_3^2\right)^2}\, e^{ixq}
\end{aligned}$$

The **D**(2)-representations are different from the Lorentz compatible position distributions of time representations as used for the quantization of the tangent Minkowski spacetime particle fields (Källen–Lehmann representations; Källen, 1952; Lehmann, 1954), for example, for a spin $\frac{1}{2}$ massive particle in a Dirac field

$$\text{particle fields}: \int_0^{\infty} dm^2\, \mu(m^2) \int \frac{d^4q}{\pi^3} \frac{i(\gamma q + |m|)}{q^2 - m^2 + io}\, e^{ixq}, \quad \mu(m^2) \geq 0$$

with probability related spectral measure $\mu(m^2)$ for the invariant of the definite unitary representations of the spacetime translations.

From the Lorentz scalar spacetime distribution with a simple energy–momentum pole one obtains

$$(\partial^2 + m^2) \int \frac{d^4q}{\pi^3} \frac{1}{q_{\mathrm{P}}^2 - m^2}\, e^{ixq} = -16\pi\,\delta(x)$$

the derivatives $\frac{\partial}{\partial m^2}$ with respect to the invariant

$$\begin{aligned}
\int \frac{d^4q}{\pi^3} \frac{\Gamma(2+N)}{\left(q_{\mathrm{P}}^2 - m^2\right)^{2+N}}\, e^{ixq} &= \begin{cases} \dfrac{\partial}{\partial \frac{x^2}{4}} \quad \vartheta(x^2)\mathcal{E}_0\left(\dfrac{x^2m^2}{4}\right), & N = -1 \\ \left(\dfrac{\partial}{\partial m^2}\right)^N \vartheta(x^2)\mathcal{E}_0\left(\dfrac{x^2m^2}{4}\right), & N = 0, 1 \ldots \end{cases}\\
&= \begin{cases} -\delta\left(\dfrac{x^2}{4}\right) + \vartheta(x^2)m^2\mathcal{E}_1\left(\dfrac{x^2m^2}{4}\right), & N = -1 \\ \vartheta(x^2)\left(-\dfrac{x^2}{4}\right)^N \mathcal{E}_N\left(\dfrac{x^2m^2}{4}\right), & N = 0, 1, \ldots \end{cases}
\end{aligned}$$

and the derivative $\frac{\partial}{\partial x}$ with respect to the Lie parameter

$$\int \frac{d^4q}{i\pi^3} \frac{q\Gamma(3+N)}{\left(q_{\rm P}^2 - m^2\right)^{3+N}} e^{ixq}$$
$$= \frac{x}{2} \begin{cases} \left(\frac{\partial}{\partial \frac{x^2}{4}}\right)^N \vartheta(x^2)\mathcal{E}_0\left(\frac{x^2m^2}{4}\right), & N = -1, -2 \\ \left(\frac{\partial}{\partial m^2}\right)^N \vartheta(x^2)\mathcal{E}_0\left(\frac{x^2m^2}{4}\right), & N = 0, 1, \ldots \end{cases}$$
$$= \frac{x}{2} \begin{cases} \delta'\left(\frac{x^2}{4}\right) - m^2\delta\left(\frac{x^2}{4}\right) + \vartheta(x^2)m^4\mathcal{E}_2\left(\frac{x^2m^2}{4}\right), & N = -2 \\ -\delta\left(\frac{x^2}{4}\right) + \vartheta(x^2)m^2\mathcal{E}_1\left(\frac{x^2m^2}{4}\right), & N = -1 \\ \vartheta(x^2)\left(-\frac{x^2}{4}\right)^N \mathcal{E}_N\left(\frac{x^2m^2}{4}\right), & N = 0, 1, \ldots \end{cases}$$

which involve the Bessel functions $\mathcal{J}_N$ with $\xi \in \mathbb{R}$

$$\mathcal{E}_N\left(\frac{\xi^2}{4}\right) = \frac{\mathcal{J}_N(\xi)}{\left(\frac{\xi}{2}\right)^N} = \left(-\frac{\partial}{\partial \frac{\xi^2}{4}}\right)^N \mathcal{J}_0(\xi) = \sum_{n=0}^{\infty} \frac{\left(-\frac{\xi^2}{4}\right)^n}{n!(N+n)!}$$
$$\left(-\frac{\xi^2}{4}\right)^N \mathcal{E}_N\left(\frac{\xi^2}{4}\right) = \left(-\frac{\xi}{2}\right)^N \mathcal{J}_N(\xi) = \left(\frac{\xi^2}{4}\right)^N \left(\frac{\partial}{\partial \frac{\xi^2}{4}}\right)^N \mathcal{J}_0(\xi)$$

The distributions for strictly negative nildimension N, that is, with a Dirac distribution on the light cone $x^2 = 0$, are no spacetime **D**(2)-representation matrix elements. One obtains for the irreducible spacetime representations the **D**(2)-functions

$$x\vartheta(x^2) \mapsto \left(m_0^2; 1, -m_3^2\right)(x)$$
$$= \int \frac{d^4q}{i\pi^3} \frac{2q}{\left(q_{\rm P}^2 - m_0^2\right)\left(q_{\rm P}^2 - m_3^2\right)^2} e^{ixq}$$
$$= x\vartheta(x^2)\left[\frac{m_0^4\mathcal{E}_2\left(\frac{x^2m_0^2}{4}\right) - m_3^4\mathcal{E}_2\left(\frac{x^2m_3^2}{4}\right)}{\left(m_0^2 - m_3^2\right)^2} + \frac{m_3^2\mathcal{E}_1\left(\frac{x^2m_3^2}{4}\right)}{m_3^2 - m_0^2}\right]$$

The neutral elements, either for **SD**(2) or **D**(1), are defined by trivial masses

$$\big(m_0^2;1,0\big)(x) = \int \frac{d^4q}{i\pi^3}\frac{2q}{\big(q_{\rm P}^2-m_0^2\big)\big(q_{\rm P}^2\big)^2}\,e^{ixq} = x\vartheta(x^2)\mathcal{E}_2\left(\frac{x^2m_0^2}{4}\right)$$

$$\big(0;1,-m_3^2\big)(x) = \int \frac{d^4q}{i\pi^3}\frac{2q}{q_{\rm P}^2\big(q_{\rm P}^2-m_3^2\big)^2}\,e^{ixq} = x\vartheta(x^2)\left[-\mathcal{E}_2\left(\frac{x^2m_3^2}{4}\right) + \mathcal{E}_1\left(\frac{x^2m_3^2}{4}\right)\right]$$

$$(0;1,0)(x) = \int \frac{d^4q}{i\pi^3}\frac{2q}{\big(q_{\rm P}^2\big)^3}\,e^{ixq} = \frac{x}{2}\vartheta(x^2)$$

9. ASSOCIATED RESIDUAL DISTRIBUTIONS

Given a residual distribution $I(q) = I_0(q)$ for an irreducible representation of a symmetric space (Lie group) G, singular distributions $\{I_N(q)\}$ using the same pole locations, but with possibly different orders, are called I-associated residual distributions. The possibly different singularity orders of the associated distributions will be characterized by integer nildimensions $N \in \mathbb{Z}$.

Associated to the Dirac distribution for an irreducible Abelian group representation are its derivatives

$$\big\{\delta^{(N)}(m-q)|N=0,1,\ldots\big\}\begin{cases} m\in\mathbb{Z} & \text{for } \textbf{irrep U}\,(1)\\ m\in\mathbb{R} & \text{for } \textbf{irrep D}\,(1)\end{cases}$$

$$\int dq\;\delta^{(N)}(m-q)\,e^{itq} = \oint\frac{dq}{2i\pi}\frac{\Gamma(1+N)}{(q-m)^{1+N}}\,e^{itq} = (it)^N\,e^{itm}$$

For the self-dual Abelian groups one has as associated distributions for the compact representations (always only where the Γ-functions are defined)

$$\left\{\frac{1}{i\pi}\frac{q\,\Gamma(1+N)}{(q^2\mp io-m^2)^{1+N}}\right\}\text{ with }\begin{cases} m\in\mathbb{Z} & \text{for } \textbf{irrep SO}(2)\\ m\in\mathbb{R} & \text{for } \textbf{irrep}^{(2,0)}\ \textbf{SO}_0(1,1)\end{cases}$$

$$\int\frac{dq}{i\pi}\frac{q\,\Gamma(1+N)}{(q^2\mp io-m^2)^{1+N}}\,e^{-ixq} = -\epsilon(x)\left(\frac{\partial}{\partial m^2}\right)^N e^{\pm i|xm|}$$

$$= \begin{cases} -\epsilon(x)\,e^{\pm i|xm|},\ \mp i\,\dfrac{x}{2|m|}\,e^{\pm i|xm|},\ldots\\ \text{for } N=0,1,\ldots\end{cases}$$

$$\pm \int \frac{dq}{i\pi} \frac{|m|\ \Gamma(1+N)}{(q^2 \mp io - m^2)^{1+N}}\ e^{-ixq} = |m| \left(\frac{\partial}{\partial m^2}\right)^N \frac{e^{\pm i|xm|}}{|m|}$$
$$= \begin{cases} e^{\pm i|xm|},\ -\frac{1 \mp i|xm|}{2m^2}\ e^{\pm i|xm|}, \ldots \\ \text{for } N = 0, 1, \ldots \end{cases}$$

and for the noncompact $\mathbf{SO}_0(1, 1)$-representations

$$\left\{ \frac{1}{i\pi} \frac{q\ \Gamma(1+N)}{(q^2 + m^2)^{1+N}} \right\} \quad \text{with } m \in \mathbb{R} \text{ for } \mathbf{irrep}^{(1,1)}\ \mathbf{SO}_0(1, 1)$$

$$\int \frac{dq}{i\pi} \frac{q\ \Gamma(1+N)}{(q^2 + m^2)^{1+N}}\ e^{-ixq} = -\epsilon(x) \left(-\frac{\partial}{\partial m^2}\right)^N e^{-|xm|}$$
$$= \begin{cases} -\epsilon(x)\ e^{-|xm|},\ -\frac{x}{2|m|}\ e^{-|xm|}, \ldots \\ \text{for } N = 0, 1, \ldots \end{cases}$$

$$\int \frac{dq}{\pi} \frac{|m|\ \Gamma(1+N)}{(q^2 + m^2)^{1+N}}\ e^{-ixq} = |m| \left(-\frac{\partial}{\partial m^2}\right)^N \frac{e^{-|xm|}}{|m|}$$
$$= \begin{cases} e^{-|xm|},\ \frac{1 + |xm|}{2m^2}\ e^{-|xm|}, \ldots \\ \text{for } N = 0, 1, \ldots \end{cases}$$

With respect to the sign of the nildimension N the residual distributions are used for

nondecomposable group representations	$\Longleftrightarrow$	$N \geq 0$
irreducible group representations	$\Longleftrightarrow$	$N = 0$
tangent representations (discussed later)	$\Longleftrightarrow$	$N \leq 0$

For compact groups strictly positive nildimensions cannot occur, they define no functions on the group

$$\text{for compact groups } N \leq 0$$

Residual distributions with strictly negative nildimensions $N = -1, -2, \ldots$ do not lead to G-representation matrix elements. They arise only for groups where the rank is strictly smaller than the dimension.

Associated to the dipole distribution of an irreducible representation of the spin group $\mathbf{SU}(2)$ and for compact representations of the position manifold $\mathbf{SD}(2)$

are the following distributions

$$\left\{\frac{1}{i\pi^2}\frac{\vec{q}\ \Gamma(2+N)}{(\vec{q}^2 \mp io - m^2)^{2+N}}\right\} \text{ with } \begin{cases} m \in \mathbb{Z} & \text{for } \mathbf{irrep}\ \mathbf{SU}(2) \\ m \in \mathbb{R} & \text{for } \mathbf{irrep}^{(2,0)}\ \mathbf{SD}(2) \end{cases}$$

$$\begin{aligned}
\int \frac{d^3q}{i\pi^2}\frac{\vec{q}\ \Gamma(2+N)}{(\vec{q}^2 \mp io - m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} &= 2\frac{\vec{x}}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial m^2}\right)^{1+N}\ \frac{e^{\pm ir|m|}}{r} \\
&= -\frac{\vec{x}}{r}\begin{cases} 2\frac{1 \mp ir|m|}{r^2}\ e^{\pm ir|m|},\ e^{\pm ir|m|}, \dots \\ \text{for } N = -1, 0, \dots \end{cases} \\
\pm \int \frac{d^3q}{i\pi^2}\frac{|m|\ \Gamma(2+N)}{(\vec{q}^2 \mp io - m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} &= \mp 2i|m|\left(\frac{\partial}{\partial m^2}\right)^{1+N}\ \frac{e^{\pm ir|m|}}{r} \\
&= \begin{cases} \mp 2i|m|\frac{e^{\pm ir|m|}}{r},\ e^{\pm ir|m|}, \\ \quad -\frac{1 \mp ir|m|}{2m^2}\ e^{\pm ir|m|}, \dots \\ \text{for } N = -1, 0, 1, \dots \end{cases}
\end{aligned}$$

and to an irreducible noncompact position **SD**(2)-representation

$$\left\{\frac{1}{i\pi^2}\frac{\vec{q}\ \Gamma(2+N)}{(\vec{q}^2 + m^2)^{2+N}}\right\} \text{ with } m \in \mathbb{R} \text{ for } \mathbf{irrep}^{(1,1)}\ \mathbf{SD}(2)$$

$$\begin{aligned}
\int \frac{d^3q}{i\pi^2}\frac{\vec{q}\ \Gamma(2+N)}{(\vec{q}^2 + m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} &= 2\frac{\vec{x}}{r}\frac{\partial}{\partial r}\left(-\frac{\partial}{\partial m^2}\right)^{1+N}\ \frac{e^{-r|m|}}{r} \\
&= -\frac{\vec{x}}{r}\begin{cases} 2\frac{1 + r|m|}{r^2}\ e^{-r|m|},\ e^{-r|m|}, \dots \\ \text{for } N = -1, 0, \dots \end{cases} \\
\int \frac{d^3q}{\pi^2}\frac{|m|\ \Gamma(2+N)}{(\vec{q}^2 + m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} &= 2|m|\left(-\frac{\partial}{\partial m^2}\right)^{1+N}\ \frac{e^{-r|m|}}{r} \\
&= \begin{cases} 2|m|\frac{e^{-r|m|}}{r},\ e^{-r|m|},\ \frac{1 + r|m|}{2m^2}\ e^{-r|m|}, \dots \\ \text{for } N = -1, 0, 1, \dots \end{cases}
\end{aligned}$$

The distributions with $N = -1$ lead to spherical waves $\frac{e^{\pm ir|m|}}{r}$ and Yukawa potentials $\frac{e^{-r|m|}}{r}$ and their derivatives which are no **SU**(2) and **SD**(2)-representation matrix elements.

The distributions associated with an irreducible **D**(2)-representation include as negative nildimension distributions

$$\frac{q\Gamma(3+N_0+N_3)}{\left(q_{\rm P}^2-m_0^2\right)^{1+N_0}\left(q_{\rm P}^2-m_3^2\right)^{2+N_3}}$$

$$\Rightarrow \begin{cases} \dfrac{q}{\left(q_{\rm P}^2-m_0^2\right)\left(q_{\rm P}^2-m_3^2\right)}, \ \dfrac{q}{\left(q_{\rm P}^2-m_3^2\right)^2}, & N_0+N_3=-1 \\ \dfrac{q}{q_{\rm P}^2-m_0^2}, \ \dfrac{q}{q_{\rm P}^2-m_3^2}, & N_0+N_3=-2 \end{cases}$$

10. RESIDUAL SUBREPRESENTATIONS

A representation of a symmetric space (Lie group) G contains representations of subspaces (subgroups) H. How does this look for residual representations?

A residual G-representation with tangent space (Lie algebra) parameters $x = (x_H, x_\perp)$

$$D^I : G \to \mathbb{C}, \quad g(x) \mapsto D^I(x) = \int d^n q \ I(q)\, e^{ixq}$$

is projected to a residual H-representation by integration $\int d^{n-s}x_\perp$ over the complementary space log G/H

$$D_H^I : H \to \mathbb{C}, \ h(x_H) \mapsto D_H^I(x_H)$$

$$\text{with } D_H^I(x_H) = \int \frac{d^{n-s}x_\perp}{(2\pi)^{n-s}} \int d^n q \ I(q)\, e^{ixq} = \int d^s q_H \ I(q_H, 0)\, e^{ix_H q_H}$$

With the integration one picks up the Fourier components for trivial tangent space forms (momenta) $q_\perp = 0$ of log G/H.

10.1. SO(2) × SO$_0$(1, 1)-Subrepresentations in Spin-Position-Representations

The **SO**(2)-subrepresentations in spin **SU**(2)-representations are given as follows

$$\textbf{irrep SU}(2) \longrightarrow \textbf{rep SO}(2), \ d^2x_\perp = d^2x_{1,2}$$

$$\int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{i\pi^2} \frac{\vec{q}\ \Gamma(2+N)}{(\vec{q}^2 \mp io - m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} = \int \frac{dq}{i\pi} \frac{q\ \Gamma(2+N)}{(q^2 \mp io - m^2)^{2+N}}\ e^{-ix_3 q}$$
$$= -\epsilon(x_3)\left(\frac{\partial}{\partial m^2}\right)^{1+N} e^{\pm i|x_3 m|}$$

$$\pm \int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{i\pi^2} \frac{|m|\ \Gamma(2+N)}{(\vec{q}^2 \mp io - m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} = \pm \int \frac{dq}{i\pi} \frac{|m|\ \Gamma(2+N)}{(\vec{q}^2 \mp io - m^2)^{2+N}}\ e^{-ix_3 q}$$
$$= |m| \left(\frac{\partial}{\partial m^2}\right)^{1+N} \frac{e^{\pm i|x_3 m|}}{|m|}$$

and the $\mathbf{SO}_0(1,1)$-subrepresentations of noncompact position $\mathbf{SD}(2)$-representations

$$\mathbf{irrep}\ \mathbf{SD}(2) \longrightarrow \mathbf{rep}^{(1,1)}\mathbf{SO}_0(1,1)$$

$$\int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{i\pi^2} \frac{\vec{q}\ \Gamma(2+N)}{(\vec{q}^2 + m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} = \int \frac{dq}{i\pi} \frac{q\ \Gamma(2+N)}{(q^2+m^2)^{2+N}}\ e^{-ix_3 q}$$
$$= -\epsilon(x_3) \left(-\frac{\partial}{\partial m^2}\right)^{1+N} e^{-|x_3 m|}$$
$$\int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{\pi^2} \frac{|m|\ \Gamma(2+N)}{(\vec{q}^2 + m^2)^{2+N}}\ e^{-i\vec{x}\vec{q}} = \int \frac{dq}{\pi} \frac{|m|\ \Gamma(2+N)}{(q^2+m^2)^{2+N}}\ e^{-ix_3 q}$$
$$= |m| \left(-\frac{\partial}{\partial m^2}\right)^{1+N} \frac{e^{-|x_3 m|}}{|m|}$$

The vector dependence $\frac{\vec{x}}{r}$ for the sphere is projected to two values $\epsilon(x_3) \in \{\pm 1\}$ for the hemispheres.

10.2. Time and Position Subrepresentations in Spacetime Representations

The energy–momentum distribution used in the residual spacetime representations is the principal value part in the decomposition of a complex distribution into imaginary and real part

$$\pm \frac{1}{i\pi} \frac{q}{\left(q^2 \mp io - m_0^2\right)\left(q_{\mathrm{P}}^2 - m_3^2\right)^2} = \underbrace{\pm \frac{1}{i\pi} \frac{q}{\left(q_{\mathrm{P}}^2 - m_0^2\right)\left(q_{\mathrm{P}}^2 - m_3^2\right)^2}}_{\text{spacetime } \mathbf{D}(2) \to \mathbb{C}}$$
$$+ \underbrace{\frac{1}{\left(m_0^2 - m_3^2\right)^2} q\delta\left(q^2 - m_0^2\right)}_{\text{tangent translations } \mathbb{R}^4 \to \mathbb{C}}$$

which is also the decomposition for the representation matrix elements of spacetime $\mathbf{D}(2)$ and its tangent space $\mathbb{R}^4$. The integrated principal value part has causal support whereas the integrated Dirac distribution for the particle pole gets both spacelike and causal support. The decomposition with respect to the two singularities

$$\begin{aligned}
&\frac{1}{(q^2-m_0^2)(q^2-m_3^2)^2}\\
&\quad=\frac{1}{(m_0^2-m_3^2)^2}\left[\frac{1}{q^2-m_0^2}-\frac{q^2-m_0^2}{(q^2-m_3^2)^2}\right]\\
&\quad=\frac{1}{(m_0^2-m_3^2)^2}\left(\frac{1}{q^2-m_0^2}-\frac{1}{q^2-m_3^2}\right)-\frac{1}{m_0^2-m_3^2}\frac{1}{(q^2-m_3^2)^2}
\end{aligned}$$

is not parallel with the representation of the factors in $\mathbf{D}(2)=\mathbf{D}(\mathbf{1}_2)\times\mathbf{SD}(2)$. The projections to representation matrix elements of the manifold factors are given by position integration for the causal group $\mathbf{D}(\mathbf{1}_2)$ and by time integration for the position manifold $\mathbf{SD}(2)$ with Cartan subgroup $\mathbf{SO}_0(1,1)$, that is by the Fourier transforms for trivial momenta $\vec{q}=0$ and trivial energy $q_0=0$ respectively

$$\begin{aligned}
&\int d^3x:\ \mathbf{irrep}\,\mathbf{D}(2)\to\ \mathbf{rep}\,\mathbf{D}(1)\\
&\int dx_0:\ \mathbf{irrep}\,\mathbf{D}(2)\to\ \mathbf{rep}\,\mathbf{SD}(2)\\
&\int d^2x_\perp:\ \mathbf{rep}\,\mathbf{SD}(2)\to\ \mathbf{rep}\,\mathbf{SO}_0(1,1)
\end{aligned}$$

where one uses

$$\begin{aligned}
&\begin{pmatrix}\int\frac{d^3x}{8\pi}\\ \int\frac{dx_0}{2}\\ \int\frac{d^2x_\perp}{4\pi}\int\frac{dx_0}{2}\end{pmatrix}\int\frac{d^4q}{i\pi^3}\frac{q\,\Gamma(3+N)}{(q_{\mathrm{P}}^2-m^2)^{3+N}}\,e^{ixq}\\
&\quad=\left(\frac{\partial}{\partial m^2}\right)^{2+N}\begin{pmatrix}\epsilon(x_0)\cos x_0m\\ 2\frac{\vec{x}}{r}\frac{1+r|m|}{r^2}\,e^{-r|m|}\\ -\epsilon(x_3)\,e^{-|x_3m|}\end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix}\int\frac{d^3x}{8\pi}\\ \int\frac{dx_0}{2}\\ \int\frac{d^2x_\perp}{4\pi}\int\frac{dx_0}{2}\end{pmatrix}\int\frac{d^4q}{\pi^3}\frac{\Gamma(2+N)}{(q_{\mathrm{P}}^2-m^2)^{2+N}}\,e^{ixq}=-\left(\frac{\partial}{\partial m^2}\right)^{1+N}\begin{pmatrix}\frac{\sin|x_0m|}{|m|}\\ 2\frac{e^{-r|m|}}{r}\\ \frac{e^{-|x_3m|}}{|m|}\end{pmatrix}$$

This leads for irreducible spacetime representations to

$$\int \frac{d^3x}{8\pi}\big(m_0^2;1,-m_3^2\big)(x) = \int \frac{dq_0}{i\pi}\frac{q_0}{\big(q_{0P}^2-m_0^2\big)\big(q_{0P}^2-m_3^2\big)^2}\,e^{ix_0q_0}$$

$$= \epsilon(x_0)\left[\frac{\cos x_0m_0-\cos x_0m_3}{\big(m_0^2-m_3^2\big)^2}+\frac{|x_0|\,\sin|x_0m_3|}{2|m_3|\big(m_0^2-m_3^2\big)}\right]$$

$$\int \frac{dx_0}{2}\big(m_0^2;1,-m_3^2\big)(x)$$

$$= \int \frac{d^3q}{i\pi^2}\frac{\vec{q}}{\big(\vec{q}^{\,2}+m_0^2\big)\big(\vec{q}^{\,2}+m_3^2\big)^2}\,e^{-i\vec{x}\vec{q}}$$

$$= -\frac{\vec{x}}{r}\left[2\frac{(1+r|m_0|)\,e^{-r|m_0|}-(1+r|m_3|)\,e^{-r|m_3|}}{r^2\big(m_0^2-m_3^2\big)^2}+\frac{e^{-r|m_3|}}{m_0^2-m_3^2}\right]$$

$$\int \frac{d^2x_\perp}{4\pi}\int\frac{dx_0}{2}\big(m_0^2;1,-m_3^2\big)(x)$$

$$= \int \frac{dq}{i\pi}\frac{q}{\big(q^2+m_0^2\big)\big(q^2+m_3^2\big)^2}\,e^{-ix_3q}$$

$$= -\epsilon(x_3)\left[\frac{e^{-|x_3m_0|}-e^{-|x_3m_3|}}{\big(m_0^2-m_3^2\big)^2}+\frac{|x_3|\,e^{-r|m_3|}}{2|m_3|\big(m_0^2-m_3^2\big)}\right]$$

The measure of the invariants for an irreducible spacetime representation

$$\text{for } \mathbf{D}(2):\ \rho\big(M_0^2,M_3^2\big)=\delta\big(M_0^2-m_0^2\big)\delta\big(M_3^2-m_3^2\big)$$

is projected to measures for the representation of the two factors. The time $\mathbf{D}(\mathbf{1}_2)$-subrepresentation with the measure

$$\text{for } \mathbf{D}(\mathbf{1}_2):\ \rho_0(m^2)=\frac{\delta\big(m^2-m_0^2\big)-\delta\big(m^2-m_3^2\big)}{\big(m_0^2-m_3^2\big)^2}+\frac{\delta'\big(m^2-m_3^2\big)}{m_0^2-m_3^2}$$

contains matrix elements of reducible nondecomposable representations for the nonparticle dipole at m_3^2.

The linear combinations occurring in the position **SD**(2)-projections of spacetime **D**(2)-representations are matrix elements of measured **SD**(2)-representations involving the difference of two Yukawa potentials

$$2\,\frac{e^{-r|m_0|}-e^{-r|m_3|}}{r}=\int_{m_0^2}^{m_3^2}dm^2\,\frac{e^{-r|m|}}{|m|}$$
$$=\int_0^\infty dm^2\vartheta\left(m^2-m_0^2\right)\vartheta\left(m_3^2-m^2\right)\frac{e^{-r|m|}}{|m|}$$

The measure for the **SD**(2)-subrepresentation reads

$$\text{for}\,\mathbf{SD}(2):\rho_3(m^2)=-\frac{\vartheta\left(m^2-m_0^2\right)\vartheta\left(m_3^2-m^2\right)}{\left(m_0^2-m_3^2\right)^2}+\frac{\delta\left(m^2-m_3^2\right)}{m_0^2-m_3^2}$$

11. RESIDUAL TANGENT DISTRIBUTIONS

The residual tangent distributions for an irreducible symmetric space (group) representation will be defined by the associated distributions with a simple pole, that is, for minimal negative nildimension $N\leq 0$, and a trivial invariant. They arise as the inverse differential operators in the Lie algebra action representing differential equations of motions.

The tangent $\mathbb{R}$ distributions for the Abelian groups have trivial nildimensions $N=0$ for the non–self-dual ones

$$\left.\begin{array}{r}\log\mathbf{U}(1):\\ \log\mathbf{D}(1):\end{array}\right\}\delta(q)\cong\frac{1}{2i\pi}\frac{1}{q},\quad\oint\frac{dq}{2i\pi}\frac{1}{q}e^{itq}=1$$

for the self-dual compact representations

$$\left.\begin{array}{r}\log\mathbf{SO}(2):\\ \log\mathbf{SO}_0(1,1):\end{array}\right\}\frac{1}{i\pi}\frac{q}{q^2\mp io},\quad\int\frac{d^1q}{i\pi}\frac{q}{q^2\mp io}e^{itq}=\epsilon(t)$$

and for the self-dual noncompact $\mathbf{SO}_0(1,1)$-representations

$$\log\mathbf{SO}_0(1,1):\ \frac{1}{i\pi}\frac{q}{q^2+o^2},\quad\int\frac{d^1q}{i\pi}\frac{q}{q^2+o^2}e^{-ixq}=-\epsilon(x)$$

For the nonabelian rank 1 spaces the residual tangent $\mathbb{R}^3$ distributions come with nildimension $N=-1$

$$\left.\begin{array}{r}\log\mathbf{SU}(2):\\ \log\mathbf{SD}(2):\end{array}\right\}\frac{1}{i\pi^2}\frac{\vec{q}}{\vec{q}^2\mp io-m^2},\quad\int\frac{d^3q}{i\pi^2}\frac{\vec{q}}{\vec{q}^2\mp io}e^{-i\vec{x}\vec{q}}=-2\frac{\vec{x}}{r^3}$$

and in the noncompact case

$$\log\mathbf{SD}(2):\ \frac{1}{i\pi^2}\frac{\vec{q}}{\vec{q}^2+o^2},\quad\int\frac{d^3}{i\pi^2}\frac{\vec{q}}{\vec{q}^2+o^2}e^{-i\vec{x}\vec{q}}=-2\frac{\vec{x}}{r^3}$$

They lead both to the Coulomb force with the Cartan subalgebra projection

$$\int \frac{d^2 x_\perp}{4\pi} 2\frac{\vec{x}}{r^3} = \epsilon(x_3)$$

The residual tangent spacetime $\mathbb{R}^4$ distributions have nildimension $N_0 + N_3 = -2$

$$\log \mathbf{D}(2)\colon \frac{1}{i\pi^3}\frac{q}{q_P^2}, \quad \int \frac{d^4 q}{i\pi^3}\frac{q}{q_P^2}\, e^{ixq} = \frac{x}{2}\delta'\left(\frac{x^2}{4}\right)$$

$$\text{with projections} \begin{pmatrix} \int \frac{d^3 x}{8\pi} \\ \int \frac{dx_0}{2} \\ \int \frac{d^2 x_\perp}{4\pi} \int \frac{dx_0}{2} \end{pmatrix} \int \frac{d^4 q}{i\pi^3}\frac{q}{q_P^2}\, e^{ixq} = \begin{pmatrix} \epsilon(x_0) \\ 2\frac{\vec{x}}{r^3} \\ \epsilon(x_3) \end{pmatrix}$$

12. DEFINING REPRESENTATIONS FOR TIME, POSITION, AND SPACETIME

Spacetime, particles, and interactions cannot be taken as separate concepts. Spacetime is known via interacting particles and the interactions of particles can be understood only in spacetime.

This connection will be translated into the mathematical language with the concept of a defining representation, familiar from Lie groups. For example, the Lie group $\mathbf{SU}(n)$ is defined by the automorphisms of a vector space $V \cong \mathbb{C}^n$ compatible with a scalar product–the linear space and the operating group merge in the concept of the defining representation.

In addition to one defining representation for some Lie groups there exist fundamental representations that reflect the rank and the number of independent invariants. For example, the Lie symmetry $\mathbf{SU}(r+1)$ one has r fundamental representations whose highest weights are basic vectors for the $\mathbb{Z}$-module with all weights. The products of a defining representation may build the fundamental ones, as in the case of $\mathbf{SU}(n)$ via the totally antisymmetric Grassmann powers of the defining vector space.

12.1. The Harmonic Oscillator—Defining a Compact Time

The irreducible time $\mathbf{D}(1)$ representation in the group $\mathbf{U}(1)$ as seen in the quantization for creation and annihilation operators $(\mathrm{u}, \mathrm{u}^*)$ of a harmonic Fermi or Bose oscillator with frequencies $\pm m \in \mathbb{R}$

$$\mathbf{D}(1) \ni e^t \mapsto e^{itm} = [\mathrm{u}^*, \mathrm{u}]_\pm(t) \in \mathbf{U}(1)$$

defines a compact model for time with the invariant $\frac{1}{|m|}$ as characteristic time unit.

The adjoint action with the Hamiltonian as the represented Lie algebra basis defines the time translations in the equations of motion[10]

$$H = m\frac{[\mathrm{u}, \mathrm{u}^*]_\mp}{2} \Rightarrow \begin{cases} \dfrac{d\mathrm{u}}{dt} = [iH, \mathrm{u}] = im\mathrm{u}, & \mathrm{u}(t) = e^{itm}\mathrm{u} \\ \dfrac{d\mathrm{u}^*}{dt} = [iH, \mathrm{u}^*] = -im\mathrm{u}^*, & \mathrm{u}^*(t) = e^{-itm}\mathrm{u}^* \end{cases}$$

The operators are $\mathbf{U}(1)$-isomorphic time orbits in the $\mathbb{C}$-isomorphic representation spaces

$$\mathrm{u}, \mathrm{u}^\star : \mathbf{D}(1) \to V, V^T \cong \mathbb{C}$$

The product representations $e^{itm_1}e^{itm_2} = e^{it(m_1+m_2)}$ generate the familiar equidistant time weights (eigenvalues, frequencies) for the quantum oscillator – $\{Zm \mid Z \in \mathbb{Z}\}$ for Bose and $\{Zm \mid Z = 0, \pm 1\}$ for Fermi – which, for the states, are projected on the positive values.

12.2. The Exponential Potential—Defining a Noncompact Position

An indefinite unitary representation of the noncompact Procrustes dilatation group $\mathbf{SO}_0(1, 1)$ for dual operators (d, d*) of Fermi or Bose type with eigenvalues $\pm m \in \mathbb{R}$

$$\mathbf{SO}_0(1, 1) \ni \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{pmatrix} \mapsto \begin{pmatrix} e^{-xm} & 0 \\ 0 & e^{xm} \end{pmatrix}$$
$$= \begin{pmatrix} [\mathrm{d}^*, \mathrm{d}]_\pm & [\mathrm{d}, \mathrm{d}]_\pm \\ [\mathrm{d}^*, \mathrm{d}^*]_\pm & \pm[\mathrm{d}, \mathrm{d}^*]_\pm \end{pmatrix}(x) \in \mathbf{SU}(1, 1)$$

defines a faithful model for the position space Cartan subgroup $\mathbf{SO}_0(1, 1)$ with the invariant $\frac{1}{|m|}$ as characteristic length unit.

The translations are implemented with the basis

$$D = im\frac{[\mathrm{d}, \mathrm{d}^*]_\mp}{2} \Rightarrow \begin{cases} \dfrac{d\mathrm{d}}{dx} = [iD, \mathrm{d}] = -m\mathrm{d}, \ \mathrm{d}(x) = e^{-xm}\mathrm{d} \\ \dfrac{d\mathrm{d}^*}{dx} = [iD, \mathrm{d}^*] = m\mathrm{d}^*, \ \mathrm{d}^*(x) = e^{xm}\mathrm{d}^* \end{cases}$$

The operators are noncompact $\mathbf{D}(1)$-isomorphic dilatation orbits in the $\mathbb{C}$-isomorphic representations spaces

$$\mathrm{d}, \mathrm{d}^* : \mathbf{SO}_0(1, 1) \to V, V^T \cong \mathbb{C}$$

[10] u Without argument means u(0), that is, for the trivial translation.

The product representations (convolutions) lead to exponentials with the eigenvalues $\{zm \mid z = 0, \pm 1\}$ for Fermi and $\{zm \mid z \in \mathbb{Z}\}$ for Bose.

A representation matrix element of the symmetric space position model $\mathbf{SD}(2) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$

$$\mathbf{SD}(2) \ni e^{-\vec{x}\vec{\sigma}} \mapsto -\frac{\vec{\sigma}\vec{x}}{r} e^{-r|m|} = \int \frac{d^3q}{i\pi^2} \frac{\vec{\sigma}\vec{q}}{(\vec{q}^2 + m^2)^2} e^{-i\vec{x}\vec{q}} = \{\psi^*, \psi\}(\vec{x})$$

with Pauli matrices $\vec{\sigma}$ defines a noncompact position with a characteristic length $\frac{1}{|m|}$ (interaction range), implemented by $\mathbb{C}^2$-valued Pauli spinor fields on the position manifold

$$\psi^A, \psi^*_A : \mathbf{SD}(2) \to V, V^T \cong \mathbb{C}^2, \quad A = 1, 2$$

The Cartan subgroup $\mathbf{SO}_0(1, 1)$ is represented by an indefinite unitary $\mathbf{SU}(1, 1)$-representation matrix element $e^{-r|m|}$.

The product representations (convolutions) add up the noncompact invariants $\{n|m| \mid n = 1, 2, \ldots\}$ in the exponential and are multiplied with spherical harmonics of degree $\{2J \mid 2J = 0, 1, 2, \ldots\}$ for the representation of the sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$.

12.3. Defining Spacetime with Two Invariants

The representation matrix element

$$\mathbf{D}(2) \ni \vartheta(x^2)x \mapsto \int \frac{d^4q}{i\pi^3} \frac{2\sigma^j q_j}{\left(q_{\mathrm{P}}^2 - m_0^2\right)\left(q_{\mathrm{P}}^2 - m_3^2\right)^2} e^{ixq} = \epsilon(x_0)\{\mathbf{\Psi}^*, \mathbf{\Psi}\}(x)$$

defines symmetric spacetime (Saller, 1997). The two invariants m_0^2 and m_3^2 characterize time and position and give units for particle masses and interaction lengths. The representation is implemented by $\mathbb{C}^2$-valued Weyl spinor fields (Heisenberg, 1967)

$$\mathbf{\Psi}^A, \mathbf{\Psi}^*_A : \mathbf{D}(2) \to V, V^T \cong \mathbb{C}^2, \quad A = 1, 2$$

It involves two conjugations—a definite $\mathbf{U}(2)$-conjugation for the time $\mathbf{D}(1)$-representation and an indefinite $\mathbf{U}(1, 1)$-conjugation for the position $\mathbf{SD}(2)$-representation. Therefore only the particle pole can be endowed with an additional asymptotic positive unitary spacetime translation representation structure by adding a real on shell contribution via $\pm\frac{1}{i\pi}\frac{1}{q^2 \mp io - m_0^2}$. A parametrization with creation and annihilation operators has to take care of the indefinite conjugation involved.

The product representations of the defining spacetime representation will give rise to product invariants that, in the case of an accompanying definite unitary conjugation, can be identified with particle masses for bound states. To carry out such a program explicitly, that is to compute a mass spectrum from the spacetime defining two invariants, the representation characteristic ratio $\frac{m_0^2}{m_3^2}$ has to be determined as well as the relevant normalization factors to be used in the eigenvalue equations for the product representation invariants.

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